Lec 7 2.4 Row echelon matrices

$x_1 + 2x_2 + 4x_3 = 2$
$2x_1 - 5x_2 + 3x_3 = 6$
$4x_1 + 6x_2 - 7x_3 = 8$

What operations can you perform w/o altering solution

Can 1) Permute equations - simply call 2nd equation first

2) Multiply equations by a constant

3) Add a multiple of one eq to the next

Same operations can be performed on rows of the AUGMENTED MATRIX

Called ERO (Elementary Row Operations)

$$\overbrace{\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix}}^{A} \overset{P_{12}}{\sim} \overbrace{\begin{bmatrix} 2 & -5 & 3 & 6 \\ 1 & 2 & 4 & 2 \\ 4 & 6 & -7 & 8 \end{bmatrix}}^{A_1}$$

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \overset{R1 = R1 + 2R2}{\sim} \underbrace{\begin{bmatrix} 5 & 8 & 10 & 14 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix}}_{A_2}$$

$A \sim A_1$

$A \sim A_2$

we say $A \sim A_1 \sim A_2$ are ROW EQUIVALENT.

Row Echelon form

$x_1 + x_2 - x_3 = 4$

$x_2 - 3x_3 = 5$

$x_3 = 2$

substitute

and solve to get the solution (BACK SUBSTITUTION)

AUG :
$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ & 1 & -3 & 5 \\ & & 1 & 2 \end{bmatrix}$$

Rules for row echelon form :

1) Any 0 rows at bottom of matrix
2) 1st nonzero element in each row is a 1.
3) leading 1 of row i is left of
" " " j
if i is above j

Ex
$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \underset{\sim}{P_{12}} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -1 & 3 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

$$R_2 = R_2 - 2R_1 \atop \sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 3 & -5 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

$$R_2 = R_2 - R_3 \atop \sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -6 & -4 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -6 & -4 \\ & 0 & 13 & 13 \\ & 0 & 9 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ & 1 & -6 & -4 \\ & 0 & 1 & 1 \\ & 0 & 1 & 1 \end{bmatrix}$$

$R_3 = R_3 / 13$

$R_4 = R_4 / 9$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Algorithm</u>

1) Start with $m \times n$ matrix. If $A = 0$ goto 7

2) Find left most non zero column

3) Put a 1 in the 1st row of left most non zero column. (called pivot position)

4) Put 0s below pivot position in the same column

5) If no non zero rows below pivot / STOP.

6) Replace A by an $m-1 \times n$ matrix (forget 1st row.) Goto 2.

Thm: Every nonzero A can be put into row echelon form.

Remark: Row echelon form non unique.

Thm: All row echelon matrices equivalent to A have same # of non zero rows.

RANK: Let $RE(A)$ be a row echelon form of A then its rank is the # of non zero rows.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 5 \\ 3 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 1 \\ 0 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$

Interpretation:

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \qquad \begin{aligned} \vec{a}_1 &= (3\ 1\ 4) \\ \vec{a}_2 &= [4\ 3\ 5] \\ \vec{a}_3 &= [2\ {-1}\ 3] \end{aligned}$$

Elementary row operation:

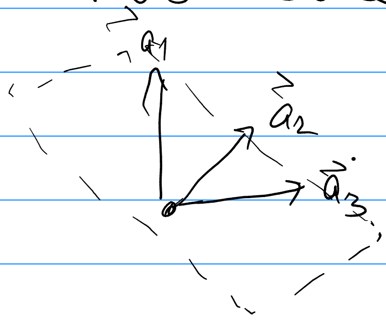
$$R1 = c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3$$

For P_{12}: $c_2 = 1$, $c_1 = 0$, $c_3 = 0$

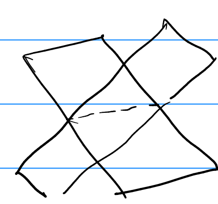
If a row is 0 then $\exists$ c_1 c_2 c_3 all not 0 st

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3 = 0$$

This means $\vec{a}_1$ $\vec{a}_2$ and $\vec{a}_3$ lie in a plane.



If 2 rows are 0 then the 3 vectors lie on two different planes



$\Rightarrow$ The lie on a line $\Rightarrow$ There is ONLY ONE vector!

Reduced Row Echelon Form (RRE)

A is in reduced row echelon form when

1) It is in row echelon form
2) Any column that has a "leading 1" has 0s everywhere else

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cancel{\checkmark} \text{ in RRE}$$

Ex

$$A \sim \begin{bmatrix} 3 & -1 & 22 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 \\ -1 & 5 & 2 \\ 2 & 4 & 24 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & -2 \\ 0 & 0 & 0 \\ 0 & 14 & 28 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$