Matrices (201)

$$\left.\begin{aligned} 3x - 8y &= -5 \\ 4x + 2y &= 6 \end{aligned}\right] \text{ find } x, y$$

$8x + 8y = 12$ add $\Rightarrow$ $11x = 7$ $\Rightarrow x = \frac{7}{11}$

and $y = \frac{1}{2}\left[6 - \frac{28}{11}\right]$

We can write this system as

$$\begin{bmatrix} 3 & -8 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} 3x - 8y \\ 4x + 2y \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Or a linear system of the form

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$ where A is the above 2×2 matrix.

Def: An $m \times n$ matrix has m rows and n columns. Its entry in the i^{th} row and j^{th} column is a number we write as A_{ij}.

A_{ij} is called an "element" of a matrix.

$m \times n$ is called "size" of the matrix

Equality of Matrices $A = B$

1) same size. $m \times n$

2) $A_{ij} = B_{ij}$. $1 \le i \le m \quad 1 \le j \le n$

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & -6 & -2 \end{bmatrix} = A \qquad B = \begin{bmatrix} -6 & 2 \\ 0 & 3 \\ -2 & -1 \end{bmatrix}$$

2×3. $\qquad 3 \times 2$ matrix.

Row and Column Vectors

Row: $1 \times n$ matrix $\qquad$ Column: $n \times 1$ matrix.

$$\vec{a} = \begin{bmatrix} -2 & \frac{1}{3} & 5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

vector notation.

Ex: if $A = \begin{bmatrix} 2 & 0 & -4 & 9 \\ 1 & 0 & 3 & 2 \end{bmatrix}$

Then A consist of the column vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

and row vectors $\begin{bmatrix} 2 & 0 & -4 & 9 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \end{bmatrix}$$

More concepts :

- Transpose of a matrix : example and definition

- Square Matrix

- lower triangular

- upper triangular

- diagonal matrix

- Skew symmetric matrices. ($A = -A^T$)

- Matrix functions : A matrix whose elements are functions

$$\begin{bmatrix} t^3 & e^{2t} & -3 \\ \log t & 1-e^{t} & \sin t \end{bmatrix}$$

Can also have complex numbers as entries.

2.2 Matrix Algebra

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 5 \\ -5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ -1 & -3 & 7 \end{bmatrix}$$

$(A+B)_{ij} = ij^{th}$ entry of $A+B$

$:= A_{ij} + B_{ij}$

Ex $(A+B)_{11} = A_{11} + B_{11} = 2-1 = 1$

Can only add matrices of the same size.

Properties $A+B = B+A$ (commutativity)

$A + (B+C) = (A+B)+C$ (associativity)

Scalar Multiplication

$A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$ $\quad 5A = \begin{bmatrix} 10 & -5 \\ 20 & 30 \end{bmatrix}$

$\uparrow$ scalar

"Multiply every entry by the scalar"

$[cA]_{ij} := cA_{ij}$

Subtraction $A - B := A + (-1)B$

Properties of scalar multiplication:

$s(A+B) = sA + sB$ (distributive property)

Zero matrix: $A+0=A$ $A-A=0$ $0A=0$

0 = all entries are 0.

Matrix multiplication:

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}_{1\times n} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} = a_1 b_1 + a_2 b_2 + \cdots \; a_n b_n$$

Called dot product: product of column and row vector.

$$\begin{bmatrix} 2 & -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \\ 4 \end{bmatrix} = 6 - 2 - 9 + 20 = 15$$

Dot product produces a number.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 1 \\ 1 & 5 \\ 0 & 6 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 3 & 6 \\ 7 & 41 \end{bmatrix}_{2\times 2}$$

Do all possible dot products!

1st row × 1st column = 11^{th} entry

i^{th} row × j^{th} column = ij^{th} entry

Notice $(2\times 3) \times (3\times 2)$ matrix gave 2×2 matrix

Ex:
$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_{3\times 1} \times \begin{bmatrix} 2 & 4 \end{bmatrix}_{1\times 2}$$ should get a 3×2 matrix

We generalize

$A = (m\times n)$ — B is $(n\times p)$

$C = AB$ — is $m\times p$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Ex $A \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ $B \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

Note BA not defined.

Properties

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

How about $AB = BA$?

$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ - $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

NOT COMMUTATIVE

Identity matrix $\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

1s along diagonal 0 elsewhere.

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

I_n = Identity matrix of size n

$$[I_n]_{ij} = \delta_{ij}$$

Properties $AI = A$, $IA = A$

let A be $m \times n$ and I be $n \times n$ (must be)

$$(AI)_{ij} = \sum_{k=1}^{n} A_{ik} I_{kj} = \sum_{k=1}^{n} A_{ik} \delta_{kj}$$

$$= A_{i1}\delta_{1j} + A_{i2}\delta_{2j} + \cdots A_{ij}\delta_{jj} + \cdots A_{in}\delta_{nj}$$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 0 & -2 \end{bmatrix}$$

Verify above property.

Properties of Transpose

$(A^T)^T = A$ $\qquad$ $(A+C)^T = A^T + C^T$

$(AB)^T = B^T A^T$

Only the last needs proof.

Lower and Upper triangular matrices

A and B both lower triangular (LT)

Then AB is also (LT)

Matrix Function Algebra

$$A = \begin{bmatrix} -2+t & e^{2t} \\ 4 & \cos t \end{bmatrix}$$

$$\frac{dA}{dt} = \begin{bmatrix} 1 & 2e^{2t} \\ 0 & -\sin t \end{bmatrix}$$ (diff term by term)

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B$$

Ex $A(t) = \begin{bmatrix} 2t & 1 \\ 6t^2 & 4e^{2t} \end{bmatrix}$

$$\int_0^1 A(t) = \begin{bmatrix} \int_0^1 2t\,dt & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 2(e^2 - 1) \end{bmatrix}$$