Lec 04

Discuss existence and uniqueness of $\frac{dy}{dx} = 3xy^{1/3}$

$y(0) = 0$

$f(x,y) = 3xy^{1/3}$ $\quad$ $\frac{\partial f}{\partial y} = \cancel{3}x\, y^{-2/3} \cdot \frac{1}{\cancel{3}}$

$= x y^{-2/3}$

This has a discontinuity at $y = 0$.

$y(0) = 0$ is a solution and so is $y = x^3$ since

$\frac{dy}{dx} = 3x^2 = 3x(x^3)^{1/3} = 3x^2$.

These two solutions meet at 0.

1.6 Integration Factors

$$\frac{dy}{dx} + \frac{1}{x} y = e^x$$

Note that 1) y has a coefficient depending on x

2) Its 1st order

Multiply this equation by x to get

$$x \frac{dy}{dx} + y = x e^x$$

Notice that $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

Then
$$x\frac{dy}{dx} + y = xe^x$$
$$\int \frac{d}{dx}(xy) = \int xe^x \Rightarrow xy = xe^x - e^x + C$$

using integration by parts.

This is the solution $y = e^x - \frac{e^x}{x} + \frac{C}{x}$

Lets generalize this idea to
$$\frac{dy}{dx} + p(x)\,y = q(x)$$

I will multiply this equation by $I(x)$
$$I(x)\frac{dy}{dx} + p\,I(x)\,y = I(x)\,q(x) \quad \text{—} ☆1$$

and I will hope that $I(x)$ is st
$$I'(x) = I(x)\,p(x) \quad \text{Then} \qquad ☆2$$

☆1 becomes
$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + I'(x)\,y \qquad (LHS)$$

Then we can integrate it as follows

$$\frac{d}{dx}\left[I(x)\,y\right] = I(x)\,q(x)$$

$$\int^{x} \frac{d}{dx}\left[I(x)y\right] dx = \int^{x} I(x)\,q(x)\,dx$$

$$\Rightarrow \quad I(x)\,y(x) = \int^{x} I(x)\,q(x)$$

So this hinges on us solving

$$\frac{dI}{dx} = I\,p(x)$$ which is seperable.

This $\log|I| = \int p(x)\,dx$

$$\Rightarrow \quad I = \pm e^{\int p(x)\,dx}$$

and we can always take $I = e^{\int p(x)\,dx}$

In the previous

$$I = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Ex $\frac{1}{x}\frac{dy}{dx} + y = e^{x^2/2}$

Bring it to standard form

$$\frac{dy}{dx} + xy = xe^{x^2/2}$$

$$I(x) = e^{\int x\,dx} = e^{x^2/2}$$

$$e^{x^2/2}\frac{dy}{dx} + x e^{x^2/2} y = x e^{x^2}$$

$$\frac{d}{dx}\left(e^{x^2/2} y\right) = x e^{x^2/2}$$

$$\Rightarrow \quad y = e^{-x^2/2}\int^x x e^{x^2/2} = e^{-x^2/2}\left(e^{x^2/2} + C\right)$$

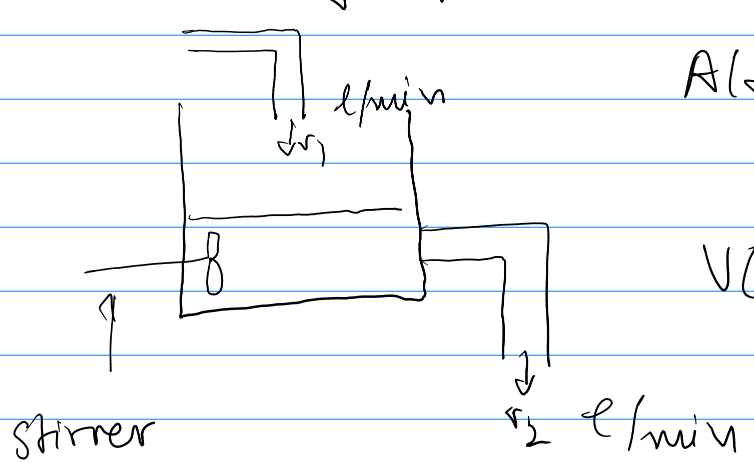
$\int^x x e^{x^2/2}$ cov $u = x^2/2$ $du = x\,dx$ $\int e^u = e^{x^2/2} + C$

So $y = 1 + Ce^{-x^2/2}$

Solve the initial value problem $y(0) = 1$.

Exercise: (Example 1.6.5 in your textbook)

1.7 Mixing problems



$A(t)$ = amount of chemical in tank. g/l

$V(t)$ = Total volume of solution

$c(t)$ = concentration in tank at time t

$$c(t) = \frac{A(t)}{V(t)}$$

r_1 = inflow rate of incoming solution l/min

c_1 = concentration of incoming solution.

Let Δt be a small time interval.

Incoming solution = $r_1 \Delta t$

New chemical added = $c_1 r_1 \Delta t$ grams

Outgoing solution = $r_2 \Delta t$

Outgoing chemical = $c(t) r_2 \Delta t$

$$\Delta V = (r_1 - r_2) \Delta t$$

$$\Delta A' = \Delta t \,(c_1 r_1 - c(t) r_2)$$

$$\frac{dV}{dt} = r_1 - r_2 \quad \Rightarrow \quad V(t) = V_0 + (r_1 - r_2) t$$

$$\frac{dA}{dt} = c_1 r_1 - \frac{A(t)}{V(t)} r_2 = c_1 r_1 - \frac{A r_2}{V_0 + (r_1 - r_2) t}$$

$$\frac{dA}{dt} + \frac{r_2}{V_0 + (r_1 - r_2) t} A = c_1 r_1$$

Lets plug in some arbitrary values to this

$r_2 = 1 \quad r_1 = 2 \quad V_0 = 4 \quad c_1 = 4$

$$\frac{dA}{dt} + \frac{1}{t+4} A = 8$$

This is a first order linear ODE

$$I(t) = e^{\int \frac{1}{t+4} dt} = e^{\log|t+4|} = |t+4|$$

May set $I(t) = (t+4)$

$$\frac{d}{dt}\left[(t+4)A\right] = 8(t+4)$$

$$A(t)(t+4) = 4(t+4)^2 + C$$

Given initial data $A(0) = 32g$

$32 \cdot 4 = 4 \cdot 16 + C \quad C = 4 \cdot 16$

$$A(t) = 4(t+4) + \frac{4 \cdot 16}{t+4}$$

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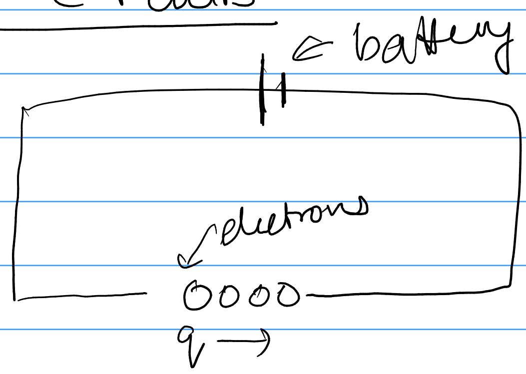
Electric Circuits



Current = rate of flow of charge = $\frac{dq(t)}{dt}$

= "electrons flowing through circuit in 1 second"

Voltage : "Ability of battery to push charges"
(12V battery)

As charges meet components of the circuit, they resists flow of charge and so ~~there are~~ typically voltage drops across them.

Kirchoff's law: Sum of voltage drops across a circuit is 0.

Resistor: Resists flow of charge. In fact

$\Delta V =$ Change in voltage $= iR$

R is resistance (ohms)

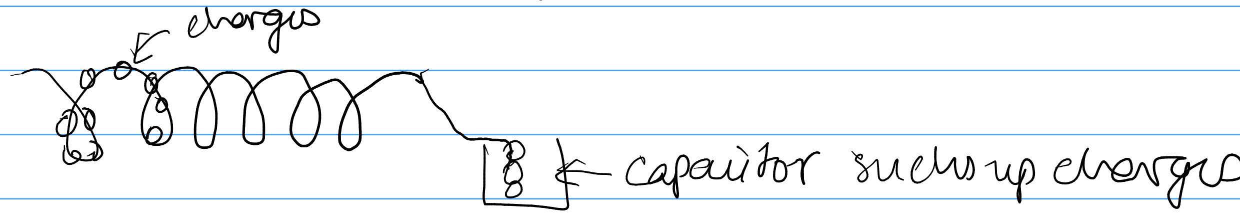
Capacitor: "Stores charges" so resists other charges flowing through it

$$\Delta V_C = \frac{1}{C} q \qquad q = \text{"Stored charge"}$$

If you open up your computer, you can see capacitors. You can discharge them using a screw driver.

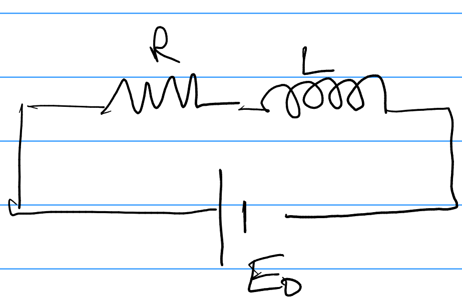
Inductor: This is a coil. It resists changes in current

$$\Delta V_L = L \frac{di}{dt}$$



How does this work again?

Ex:



Suppose battery has an alternating source

$E(t) = E_0 \cos(\omega t)$ where ω is the frequency.

By kirchoff $E(t) = iR + L\frac{di}{dt}$

$$\frac{di}{dt} + \frac{iR}{L} = E_0 \cos(\omega t)$$

$$\frac{d}{dt}\left[e^{R/L\, t}\, i\right] = E_0 \cos(\omega t)\, e^{R/L\, t}$$

$$e^{R/L\, t}\, i(t) = E_0\left[\cos(\omega t)\frac{R}{L} e^{R/L\, t} - \omega\int \sin(\omega t)\, e^{R/L\, t}\right]$$

$$I = \int^t \cos(\omega t)\, e^{R/L\, t} = \left[\cos(\omega t)\frac{R}{L} e^{R/L\, t} + \omega\int \sin(\omega t)\, e^{R/L\, t}\right]$$

$$= \frac{\cos(\omega t)}{a} e^{R/L\, t} + \frac{\omega \sin}{a^2}(\omega t)\, e^{R/L\, t} - \frac{\omega^2 I}{a^2}$$

$$\Rightarrow I = \frac{a\cos(\omega t) + \omega \sin(\omega t)}{a^2 + \omega^2} e^{at}$$

where $a = R/L$

Similarly we can have RLC circuits, RC circuits and LC circuits.

$$i(t) = \frac{E_0}{L(a^2+\omega^2)}\left(a\cos\omega t + \omega\sin\omega t - \underbrace{a e^{-at}}_{\text{transient part}}\right)$$

$$i_s(t) = \frac{E_0}{L(a^2+\omega^2)}\left(a\cos\omega t + \omega \sin \omega t\right)$$

$$= \frac{E_0}{L\sqrt{a^2+\omega^2}} \cos(\omega t - \phi)$$

which shows that the RL circuit "shifts" the phase of the periodic input.