Lec 3

Solve

$$(e^y + 4y^2)\frac{dy}{dx} = 9xe^{3x}$$

" Illegal operation "

$$\int^y (e^y + 4y^2)\,dy = \int^x 9x e^{3x}\,dx$$

$$e^y + \frac{4y^3}{3} = \frac{9xe^{3x}}{3} - \int 3e^{3x} + C$$

$$e^y + \frac{4y^3}{3} = 3xe^{3x} - e^{3x} + C$$

Claim: This is a solution of the form

$$F(x,y) = e^y + \frac{4y^3}{3} - 3xe^{3x} + e^{3x} + C = 0$$

Pf:
$$\frac{d}{dx}F(x,y(x)) = e^y\frac{dy}{dx} + 4y^2\frac{dy}{dx} - 3e^{3x} - 9xe^{3x} + 3e^{3x}$$
$$= 0$$

$$\Rightarrow \frac{dy}{dx}\left(e^y + 4y^2\right) - 9xe^{3x} = 0$$

General form of SEPERABLE DE

$$q(y)\frac{dy}{dx} = p(x)$$

This has solution

$$\int^{y} q(y)\,dy = \int^{x} p(x)\,dx$$

The proof is using the chain rule as above.

Examples: Find all solutions to

$$y' = -2y^2 x$$

Note: $y = 0$ is a solution since $y' = 0$

$$\frac{y'}{y^2} = -2x$$

$$\int^{y} \frac{1}{y^2}\,dy = \int^{x} -2x\,dx + c \text{ is a solution, so}$$

$$-\frac{1}{y} = -x^2 + c \Rightarrow y = \frac{1}{x^2 - c}$$

is a solution (for various values of c)

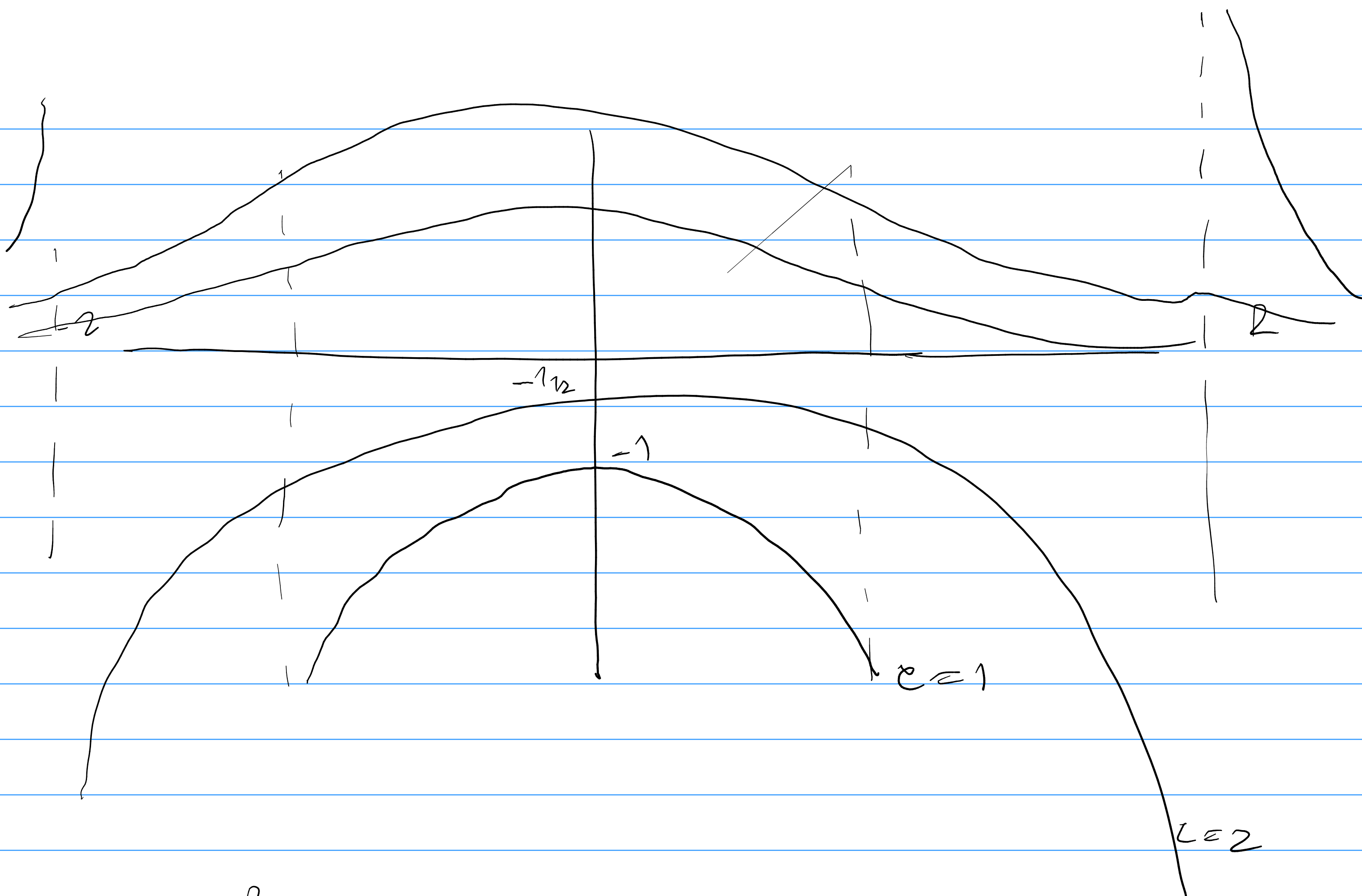
Assume first that $c \geq 0$

Note that when $x = \pm c$ y is undefined.

Also note $x \to c^- \Rightarrow y \to -\infty$ and $x \to c^+$, $y \to +\infty$

So when $-c < x < c$ differentiate $y' = \frac{-1}{(x^2 - c)^2} 2x$

So y' is -ve when $x > 0$ and $y' \geq 0$ when $x \leq 0$. When $x = 0$, $y' = 0$ and $y = \frac{1}{-c}$



For $|x|^2 > c$ y is +ve and decreasing in x

What happens when $c < 0$ Then

$$y(x) = \frac{1}{x^2 - c} \quad \text{and} \quad y > 0.$$

Then $y' = -\frac{1}{(x^2-c)^2} 2x$ again has a critical point at $x = 0$

This is a maximum.

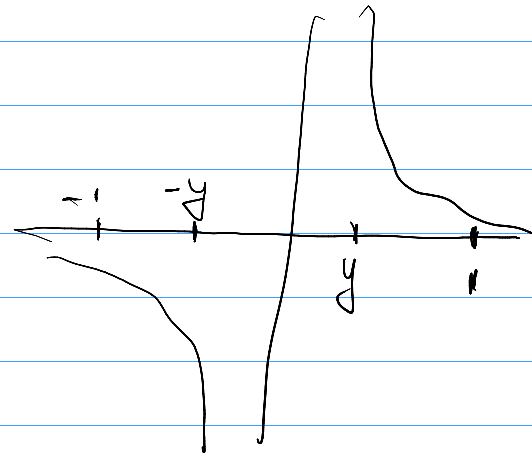
Again note that I have drawn slope fields so that they do not cross. This is a property of the existence and uniqueness theorem.

That we will review soon.

IMPORTANT NOTE: We ensured that $y \neq 0$ before dividing both sides of the DE

The Integral of $\frac{1}{y}$:

Consider $\int_{-1}^{-y} \frac{1}{t}\,dt$



We know $\int_{y}^{1} \frac{1}{t}\,dt = \log t \Big|_{y}^{1} = -\log y$

$\Rightarrow \int_{1}^{y} = \log y = \log|y|$

(which is +ve when $0<y<1$) so the area under the when $t<0$ must be negative and so

$$\int_{-1}^{-y} \frac{1}{t}\,dt = +\log y = \log|y|$$

So the thing to remember here is that

$$\int^{y} \frac{1}{t}\,dt = \log|y| + C$$

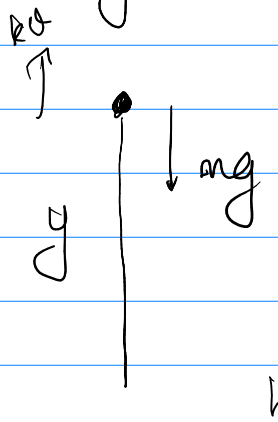
since you cannot take the log of a negative number

Now use this to try Ex 1.4.6 in your text book

$$\frac{1}{y}\,dy = -2x\,dx$$

<u>Example</u> let an object of mass m fall from rest

Assume that the "air resistance" is proportional to the apple's velocity and solve for the height of the apple as a fn of time.



The air resistance is kv.

Let $h(t)$ = height. But lets measure

h downwards

Then $a = \frac{d^2h}{dt^2}$ $\quad v = \frac{dh}{dt}$

$$m\frac{d^2h}{dt^2} = mg - k\frac{dh}{dt} \qquad k>0.$$

We dont know how to solve 2nd order ODE as yet.

$$v = \frac{dh}{dt}$$

Then $m\frac{dv}{dt} = mg - kv$

$$\frac{m\frac{dv}{dt}}{kv - mg} = -1$$

This is a separable equation. where

$$f(v)\frac{dv}{dt} = -1 \qquad f(v) = \frac{1}{\frac{k}{m}v - g}$$

How to integrate $\displaystyle\int \frac{1}{\frac{k}{m}v - g}\,dv$

Make a COV: $u = \frac{k}{m}v - g$ $\quad du = \frac{k}{m}\,dv$

$$\int \frac{k/m \, du}{u} = \frac{k}{m} \log |u|$$

$$= \frac{k}{m} \log \left| \frac{k}{m} v - g \right|$$

→ The solution is

$$\frac{k}{m} \log \left| \frac{k}{m} v - g \right| = -t + c$$

$$\log \frac{|kv - mg|}{m} = -\frac{m}{k} t + c \qquad \left(\log \frac{a}{b} = \log a - \log b \right)$$

$$|kv - mg| = e^{-m/k \, t + c} = e^{c} e^{m/k \, t}$$

$$= C e^{-m/k \, t} \qquad (\text{relabeling } e.)$$

Now think about whats going to happen:

v is going to increase (starting from 0)

So $kv - mg$ is going to be -ve at least until $kv \geq mg$

So we may as well write

$$-kv + mg = Ce^{-m/k \, t}$$

lets use the initial condition $v(0) = 0$ to solve

for C. $-0 + mg = C$

$$-kv + mg = +mg \, e^{m/k \, t}$$

$$v = \frac{mg}{k} \left[1 - e^{-m/k \, t} \right]$$

So v approaches $\frac{mg}{k}$ from below as $t \to \infty$,

Or in other words $v \leq \frac{mg}{k}$

So this is called the terminal velocity

* What if v starts at a velocity LARGER than terminal velocity? Good HW / Exam problem.

See textbook for pic of $v(t)$.

what to do next?

$$v(t) = \frac{mg}{k}\left[1 - e^{-k/m\, t}\right]$$

$$\int_0^t \frac{dh}{dt} = \int_0^t \frac{mg}{k}\left[1 - e^{-k/m\, t}\right]$$

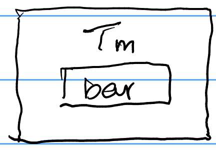
$$h(t) = \frac{mg}{k}\left[t + \frac{k}{m}\left[e^{-k/m\, t} - 1\right]\right]$$

Newton's Law of cooling A hot metal bar at 350F is in a room at 70F

After 10 minutes the temperature of the bar is at 210F

1) Find temp after 4mins

2) Time required for bar to cool to 100F



$$\frac{dT}{dt} \text{ (change in temp) } = -k\,(T - T_m)$$

$T - T_m$ is positive. So Temperature must reduce and therefore we have a negative sign.

$$\frac{1}{(T-T_m)}\frac{dT}{dt} = -k \qquad \text{(This is seperable)}$$

$$\int \frac{dT}{T-T_m} = \int -k\,dt$$

Let $u = T - T_m \qquad du = dT$

$$\int \frac{du}{u} = -kt + c \quad \Rightarrow \quad \log u = -kt + c$$

$$u = T - T_m = e^{-kt+c} \quad \Rightarrow \quad T - T_m = Ce^{-kt}$$

at $t = 0$, $T(0) = T_0 \quad \Rightarrow \quad T_0 - T_m = C e^{-k0} = C$

$$\Rightarrow \quad T - T_m = (T_0 - T_m)\,e^{-kt}.$$

$T_0 = 350 \qquad$ At $t = 2$, $T(t) = 210$

$T_m = 70$

$$\Rightarrow \quad (210 - 70) = (350 - 70)\,e^{-k2}$$

To find $\quad (T(4) - T_m) = (T(0) - T_m)\,e^{-k4} \quad \leftarrow$ A1

$$e^{-k4} = \left(e^{-k2}\right)^2$$

$$\Rightarrow \quad (T(4) - T_m) = (T(0) - T_m)\left(\frac{210-70}{350-70}\right)^2$$

Time required to cool to 100 f

So $T(t) = 100$, $(100 - 70) = (350-70)\, e^{-kt}$ ★2

From ★1 $\quad -k2 = \log \dfrac{(210-70)}{(350-70)}$

$$\Rightarrow \quad -k = \frac{1}{2}\log\frac{(210-70)}{350-70}$$

$\Rightarrow$ (from ★2)

$$-kt = \log\frac{(100-70)}{350-70}$$

$$\Rightarrow \quad t = \frac{\log\frac{(100-70)}{350-70}}{\frac{1}{2}\log\frac{(210-70)}{(350-70)}}$$

From section 1.3

Existence and uniqueness theorem.

Let $\dfrac{dy}{dx} = f(x,y)$

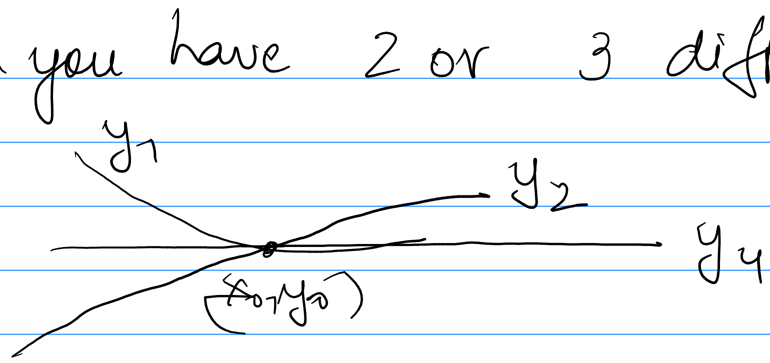
and suppose f is continuous on a rectangle.

I tell you $y_0 = y(x_0)$

Is there a function $y(x)$ such that

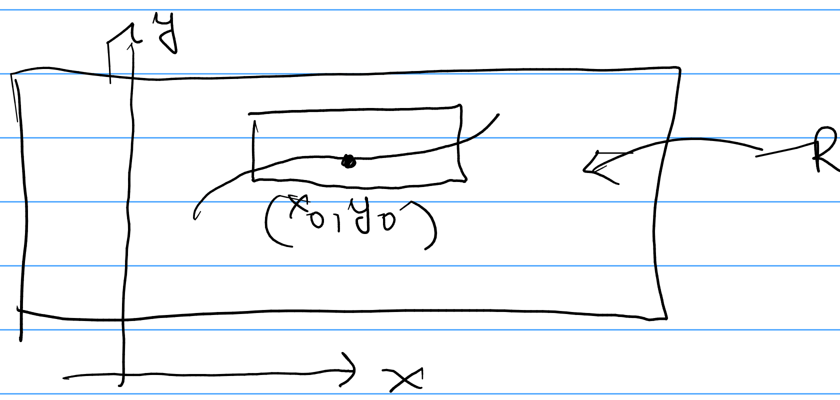
$$y(x_0) = y_0$$

Can you have 2 or 3 different solutions



The answer is : 1) $\exists$ a solution in a smaller interior box

2) There is ONLY one solution!

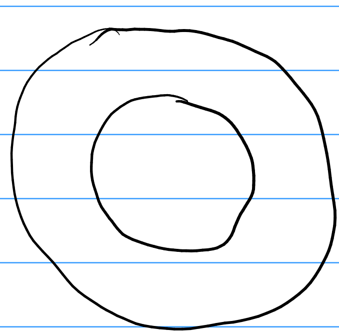


Consequence: Solution curves CANNOT cross or COALESCE.

From section 1.1

Consider the family of curves given by

$F(x,y,c) = 0$ $\quad$ $F(x,y,c) = x^2 + y^2 - c$



I want to find a a family of curves

$G(x,y,k)$ st every curve ~~is~~ orthogonal

TO ALL CURVES IN $f(x,y,c)=0$

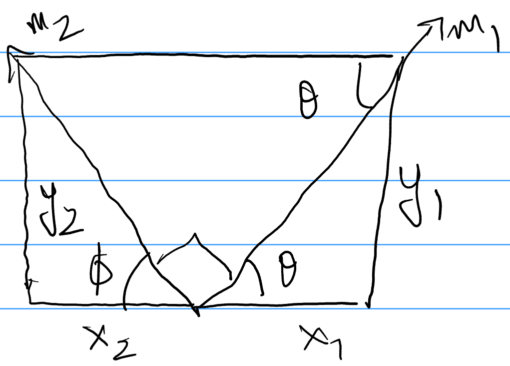
Property: if you have two lines with slopes m_1 and m_2 st they are orthogonal then

$$m_1 \cdot m_2 = -1$$

If $F(x,y,c)=0$ is a curve and has slope m_1 at point (x,y). Then the orthogonal curve is given by

$$\frac{dy}{dx} = -\frac{1}{f(x,y)}$$

($m_1 = f(x,y)$ is the slope at each point (x,y))



$\tan(\theta+90) = -\cot(\theta)$

$m_1 = \frac{y_1}{x_1}$ $\quad$ $y_2 = y_1$ by construction

$$\text{Area} = y_1(x_1+x_2) = \frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$

$$y_1x_1 + y_2x_2 = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$

$$\Rightarrow \cancel{y_1^2x_1^2} + \cancel{y_1^2x_2^2} + 2y_1^2x_1x_2 = x_1^2x_2^2 + \cancel{x_1^2y_1^2} + \cancel{y_1^2x_2^2} + y_1^2y_2^2$$

$$\Rightarrow (x_1x_2 - y_1^2) = 0 \Rightarrow x_2 = \frac{y_1^2}{x_1}$$

account for sign of x_2.

So $m_1 m_2 = -\frac{y_1}{x_1} \frac{y_2}{x_2} = 1$

$x^2+y^2+c=0$ $\quad 2x + \frac{dy}{dx} 2y = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ $\quad \Rightarrow$ slope at $(x,y) = -\frac{x}{y}$

So orthogonal curves are $\frac{dy}{dx} = \frac{y}{x}$

(seperable)

$\log y = \log x + C$

$\Rightarrow y = Cx$ (a straight line)