165 Notes

Go over syllabus.

Mention HW, webwork, recitations.

Disability, honesty.

Gravity example one

Will use $F = ma$.

$F = mg$ for object in free fall

a = acceleration = "2nd deriv of displacement"

Then $\cancel{m} \dfrac{d^2y}{dt^2} = \cancel{m} g$

In metric $g = 9.8 m/s^2$

So our 1st diffeq is

$$\frac{d^2y}{dt^2} = g$$

INITIAL CONDITIONS

$$y(0) = 5 \quad \text{and} \quad \frac{dy}{dt}(0) = 2$$

Can you interpret this?

Initial value problems

$$\int_0^t \frac{d^2y}{dt^2}\, dt = \int_0^t g$$

$$\frac{dy}{dt}(t) - \frac{dy}{dt}(0) = g(t-0)$$

using $\displaystyle\int_0^t f'(s)\,ds = f(t) - f(0)$

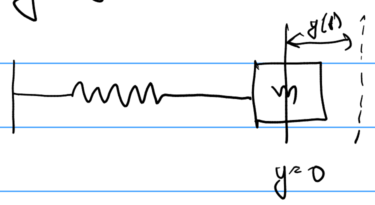
Repeat to get

$$y(t) - 5 = \frac{1}{2} g t^2$$

Question: When will the mass hit the ground?

Spring problem: Use Hooke's law

Spring force $F = -ky$



$y=0$ is equilibrium position.
$y(t)$ is displacement from equilibrium.

Use $F = ma$

$$-ky = m\frac{d^2y}{dt^2}$$

Question: Why the -ve sign?

Terminology: SIMPLE HARMONIC OSCILLATOR

$$\frac{d^2y}{dt^2} + \frac{m}{k}y = 0 \equiv y'' + \omega^2 y = 0, \quad \omega = \sqrt{\frac{m}{k}}$$

Initial conditions $y(0) = y_0 \quad y'(0) = v_0$

NOTATION: $\frac{dy}{dt} \equiv y' \equiv \dot{y}$. In general $\frac{d^n y}{dt^n} = y^{(n)}(t)$

You may verify that

$A\cos(\omega t + \phi)$ is a solution.

$$y' = -\omega A \sin(\omega t + \phi), \quad y'' = -\omega^2 A \cos(\omega t + \phi)$$

$$\Rightarrow \quad y'' + \omega^2 y = 0$$

Note two arbitrary constants A, ϕ for 2 initial data

Note linearity: If $A \cos \omega t$ is a solution.

So is $B \sin \omega t$

Thus $y = \underbrace{A \cos \omega t}_{\text{sol 1}} + \underbrace{B \sin \omega t}_{\text{sol 2}}$ also a solution

Again note 2 constants.

Newtons law of Cooling

$\boxed{\begin{array}{c}\text{House}\\ T(t)\end{array}}$ $\quad T_m$ = outside temperature.

Newton says $$\frac{dT(t)}{dt} = -k\left(T(t) - T_m\right)$$

why this sign

Rearrange to get $$\frac{dT}{dt} + kT = kT_m$$

T_m we may assume is more or less constant

(not in Rochester though)

So verify

$$T(t) = T_m + C e^{-kt}$$

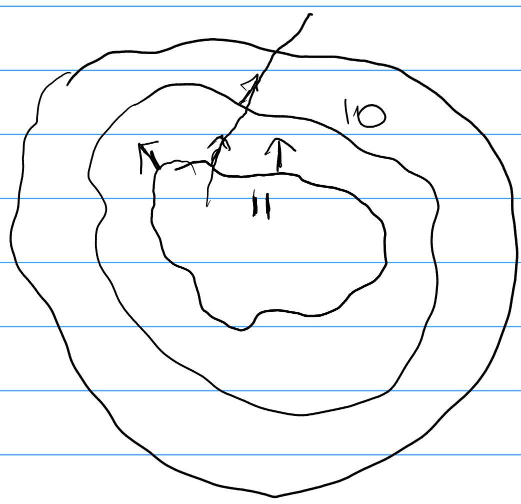
where C is an arbitrary constant.

How do we determine C?

Orthogonal Curves

Now suppose we want to determine microscopic details of heat "flow", how do we do so?

One way to model this is to say that heat flows in a normal direction to an isotherm.



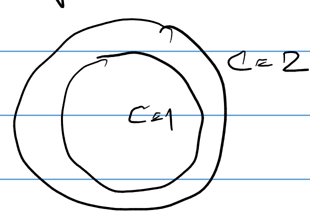
So by definition heat flow lines are ORTHOGONAL to the ISOTHERMS.

ORTHOGONAL: meets at 90°

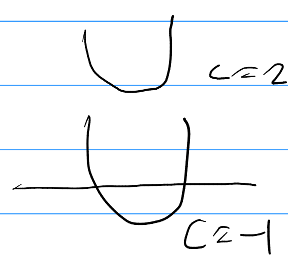
How to define a family of curves:

1) $x^2 + y^2 = c$

family of circles



2) $y^2 = x + c$



family of parabolas.

In general $F(x, y, c) = 0$ for different c.

$y_c(x)$ is an orthogonal curve if

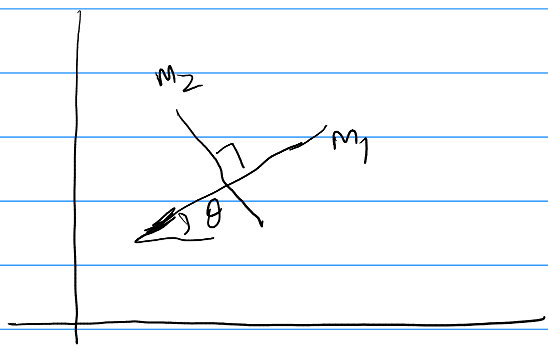
$\frac{dy}{dx}$ is at a 90° angle to the tangent

to $f(x, y, c) = 0$.

Recall that two curves are normal at (x,y)

if their tangent lines satisfy

$$m_1 \cdot m_2 = -1$$



$\tan\theta = m_1$

$\tan(\theta+90) = m_2$

$$\frac{\sin(\theta+\pi/2)}{\cos(\theta+\pi/2)} = \frac{\cos\theta}{-\sin\theta}$$

So $\tan(\theta+\pi/2) = -\cot\theta$

$$= -\frac{1}{\tan\theta} \Rightarrow \tan(\theta+\pi/2)\cdot\tan\theta = -1$$

The tangent to $F(x,y,c)$ at (x,y) we call $f_c(x,y)$. Then we need

$$\frac{dy_c}{dx} = -\frac{1}{f_c(x,y)}$$

We will illustrate with an example.

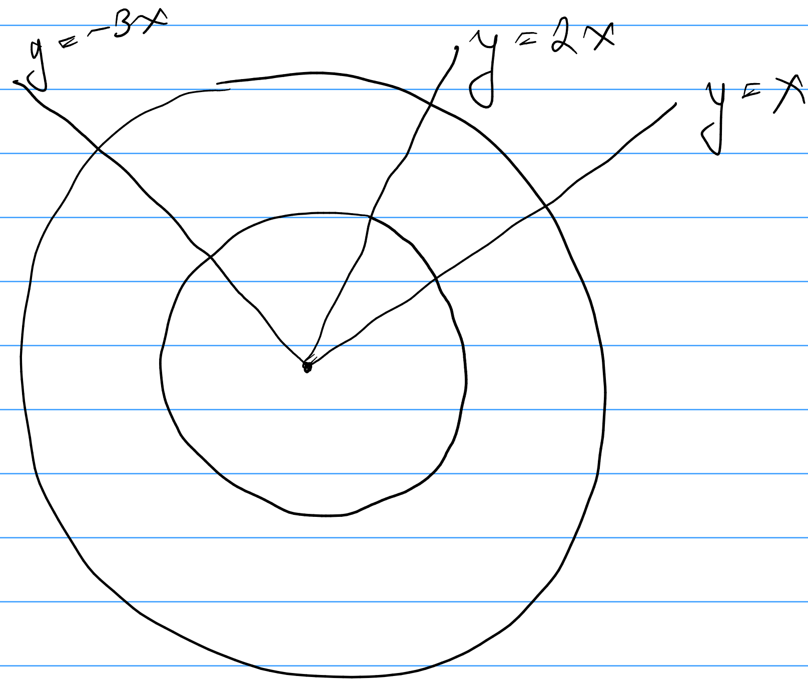
i) $x^2+y^2=c$, $\quad 2x + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} = m_2$$

$$\frac{dy}{dx} = m_1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\frac{-x}{y}} = \frac{y}{x}$$

A solution to this equation is

$y = kx$ for any constant k. This will be orthogonal to $x^2 + y^2 = c$ for all c



Do example 1.1.1 on the textbook for HW.

1.2 Basic ideas and terminology

Examples

$$\frac{dy}{dx} + y + x^2 = 0$$

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

Linear

$$\sin\left(\frac{dy}{dx}\right) + x^2 = 0$$

Nonlinear

General linear DE

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_n(x) = 0$$

Linear because no powers higher than 1 or any nonlinear functions of $y, y^{(1)}, \ldots y^{(n)}$

appear.

Examples of linear fns of y : $2y + 3x^3$

" " " $y^{(1)}$: $a y^{(1)} + b$

Examples of nonlinear fns of y : $x + \cos(y)$

ORDER : Highest derivative of y appearing in a DE.

Ex : $y'' + ky = 0$ has order 2

$y'' + x^2 y' + (\sin x) y = e^x$ is a LINEAR DE of ORDER 2

Solutions : A function $y = f(x)$ solves a DE if it satisfies the DE for all relevant values of x.

Ex : $y'' + y = 0$ has A solution $y(x) = \sin x$ on $(-\infty, \infty)$

another is $y(x) = \cos x$, another is $y(x) = a \sin x + b \cos x$

Ex : $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ only makes sense on $x > 0$

So ex : $y(x) = \frac{1}{2}\sqrt{x}$ is a solution on $(0, \infty)$

IMPLICIT FORM SOLUTIONS

Sometimes we will not be able to write

$$y = f(x), \text{ or } y - f(x) = 0 \quad \text{as a solution}$$

Note that if I write $F(x,y) = y - f(x)$ then we have a solution of the form $F(x,y) = 0$

Ex: $\dfrac{dy}{dx} = \dfrac{-x}{y}$ has a solution

$$x^2 + y^2 = 4. \quad \text{Here } F(x,y) = x^2 + y^2 - 4.$$

Why?

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(4) \quad \Leftrightarrow$$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

We can also write

$x^2 + y^2 = 4$ as $y = \pm\sqrt{4 - x^2}$ giving 2 solutions in EXPLICIT form.

Ex: $\dfrac{dy}{dx} = \dfrac{1 - y\cos(xy)}{x\cos(xy) + 2y}$

To verify $\sin(xy) + y^2 - x = 0$ defines a solution

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}\sin(x\,y(x))$$ Chainrule

$$= \cos(x\,y(x))\left[1\,y(x) + x\frac{dy}{dx}\right]$$

Thus $\frac{d}{dx}\left[\sin(xy) + y^2 - x = 0\right]$

$\Leftrightarrow$ $\cos(xy)\left[y + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} - 1 = 0$

$\Leftrightarrow$ $\frac{dy}{dx}\left[x\cos(xy) + 2y\right] = 1 - y\cos(xy)$

$\Leftrightarrow$ Gives original DE

GENERAL SOLUTION For Linear DEs we can get things called "general solutions" or forms FOR ALL possible solns.

$$\frac{d^2y}{dx^2} = 12x$$

$$\int^x \frac{d^2y}{dx^2}\,dx = \int^x 12x\,dx$$ (indefinite integral)

$\Leftrightarrow$ $\frac{dy}{dx} + c_1 = \frac{12x^2}{2} + c_2$ (combine $c_2 - c_1 = a$)

$$\int \frac{dy}{dx}\,dx = \int 6x^2 + a$$

$\Leftrightarrow$ $y = \frac{6x^3}{3} + ax + b$

Different values of a and b determine different solutions!

Question: How many total solutions are there?

$(y')^2 + (y-1)^2 = 0$ has the ONLY solution $y = 1$.

Ex 1.2.12 Find the general solution for

$y'' = e^{-2x}$ lets do it in our heads

$y' = -\frac{e^{-2x}}{2} + a$

$y = \frac{e^{-2x}}{4} + ax + b$