lec 5a Some historical review.

So far we have seen some brief historical review, and a couple of important theorems:

1) Subadditivity, Fekete's lemma
2) Existance of time constant
3) Simple comparison inequalities for the time constant.
4) Jorge will show us the subadditive ergodic theorem.

Now I want to return to the history, starting with Eden (1961). He wants to study Morphogenis (morpho- "form", genesis- "creation")

You have an individual cell. It has some "information" inside it. Say its DNA code (by then genes (1944) and the existence of DNA (Watson, Crick, $\underbrace{\text{Wilkins and Franklin}}_{\text{X-ray crystallography}}$) (1950s).

So this cell divides and it is only influenced by its random environment of nutrients and other cells. How does it achieve its final shape? How the heck does a single brain cell divide and form a brain? Amazing, right?

Eden wanted to study this question, asked by Alan Turing (1952) ("information theory") using a Toy model.

Let us read the intro starting in paragraph 2.

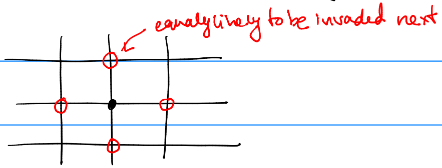
- Even twins are not exactly identical, so must be influenced by environment.

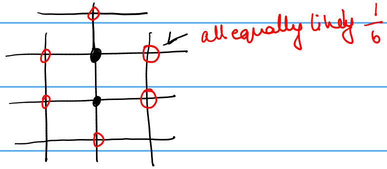
  (Fingerprints of twins, armadillo scale counts, skin color patterns of cows)

More influence of environment: growth of chicken cells into a kidney tubule or a feather.

Turings model was 1D (Does someone want to read it)

Eden's model was in 2D. (Pg 9 of his paper)





Biological counterpart: 2D growth of cells, where growth only happens at the periphery.

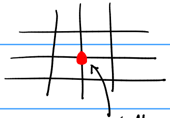
- Ulva lactuca, common sea lettuce.
- Prasiola stipitata.

On page 234(12) there a some pictures he generated. Looks pretty cool right? He claims that its a ball.

Its worth reading his paper in some detail.

Richardson (1973)

He was inspired by Williams-Bjerknes process. On the square lattice.



cell.

"The hazard function" for the time to cell division is constant.

When cell division occurs, it picks a neighbor at random. If this neighbor is white, it will turn red.

random clock rings in $[t, t+\Delta t]$

Hazard function: $h(t) = \dfrac{f(t)}{\int_t^\infty f(y)\,dy}$ "=" $\dfrac{P(T=t)}{P(T>t)}$

The only distribution with constant hazard fn is

$h(y) =$ "conditional density of dying now given that you have survived up to now."

$f(y) = \lim_{\Delta\downarrow 0} \dfrac{F(y+\Delta)-F(y)}{\Delta}$

$$h(y) = \lim_{\Delta\downarrow 0} \frac{F(y+\Delta)-F(y)}{\Delta\, P(T>y)} = \lim_{\Delta\downarrow 0} \frac{P(y<T\le y+\Delta)}{\Delta\, P(T>y)}$$

$$= \lim_{\Delta\downarrow 0} \frac{P(y<T\le y+\Delta \mid T>y)}{\Delta}$$

$P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

Thats why this is like a conditional density. But it's not a real conditional density since it does not integrate to 1.

If $T \sim Exp(\lambda)$

$$h(y) = \frac{\boxed{\lambda e^{-\lambda y}}}{\underbrace{\boxed{e^{-\lambda y}}}_{P(T>y)}} = \lambda \quad \text{(constant)}$$

Thus, attach iid edge weights to the lattice Exp(1) and at the time at which one of the exponential clocks on the edges fires, grow the cluster to that edge.

$X = \min\{T_1, \dots, T_n\}$ Suppose $T_k \neq \underbrace{\operatorname{argmin}\{T_1, \dots, T_n\}}_{=: i_0}$

$$P(T_k > X+u \mid k \neq i_0) = P(T_k > X+u \mid T_k > X)$$

$$= \int_0^\infty P(T_k > t+u \mid T_k > t)\, \lambda e^{-\lambda t}\, dt$$

$$= \int_0^\infty \frac{e^{-\lambda(t+u)}}{e^{-\lambda t}}\, \lambda e^{-\lambda t} = e^{-\lambda u}$$

So by the memoryless property, we can just replace all the T_k every time a vertex is turned red by new independent $\tilde{T}_k$. This is called 'restarting the exponential clocks.

So here's the punchline: The Eden model is just FPP with exponential edge wts, but we IGNORE the times at which vertices are added to the growing cluster.

Richardson proved the shape theorem (in meas)

$$P\big((1-\varepsilon)\mathcal{B} \subset \mathcal{B}_n \subset (1+\epsilon)\mathcal{B}_n \big) \xrightarrow{n\to\infty} 1$$

Eden's conjecture is kind of boring, but Richardson proves it.

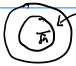
SKIP

(Eden's conjecture says that the fraction of configurations with n cells C_n likely to be generated tends to 0.)

C_n = # of configs with n cells containing the origin.

$\geq 2^n$ (random walk that goes up or right)

The config with n cells that is actually generated will be contained inside two balls of radius $(k-\epsilon)\sqrt{n}$ and $(k+\epsilon)\sqrt{n}$.



# of cells $= 2\pi k\sqrt{n}\,\epsilon$

Total # of configs $\leq 2^{2\pi k\sqrt{n}\,\epsilon}$

Thus

$$\frac{2^{2\pi k\sqrt{n}\,\epsilon}}{2^n} \longrightarrow 0$$

On page 13, he states:

"As p increases, the roughness of the boundary seems to increase. As the roughness of the boundary increases, the natural norm imposed by the lattice breaks down, and the Euclidean norm asserts itself. It would be interesting to have an explanation for this phenomenon.

He only goes to $t = 50$ (He does geometric FPP)

Richardson's conjecture is still open (Is Exponential GPP a ball?)

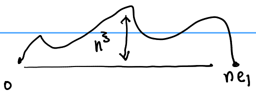
In the meantime, in the physics world

Kardar - Parisi - Zhang introduce KPZ scaling etc (1986) Phy Rev. Lett.

Zabolitsky - Stauffer simulate the Eden model (1986) Phys. Rev. A.

The general conjecture was that in 2D surface growth

$$\sqrt{\mathrm{Var}\,(T(0, n e_1))} \approx n^{\chi} \leftarrow \text{fluctuation exponent}$$



$\chi = \frac{1}{3}$ and $\xi = \frac{2}{3}$ in $d=2$

$\underbrace{2\xi - 1 = \chi}_{\text{KPZ relationship.}}$ in general d

The 90s saw a lot of mathematical activity (independent of physics)

but no simulation studies had been performed.

This gap was filled finally in 2013 (Alm & Deijfen)

But even now their simulations seem small compared

to what one would do on a modern set of GPU.

↳ Could ask Sreepathi Pai at UR if this is a possibility.

Their main question is the following:

Given cdfs F and $\tilde{F}$, and corresponding FPP models.

When is $\mu_{\tilde{F}} \leq \mu_F$?

We know vBK criterion. If $F(0) < p_c$ and

$F <_{var} \tilde{F}$ $\quad \left(\int \phi \, d\tilde{F} \leq \int \phi \, dF \quad \forall \text{ concave increasing } \phi \right)$

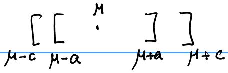
then $\mu_F < \mu_{\tilde{F}}$ (if $F \neq \tilde{F}$)

In particular if

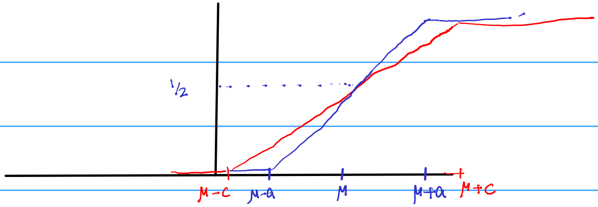
$F = \text{Unif}\,[\mu - a, \mu + a]$

$\tilde{F} = \text{Unif}\,[\mu - c, \mu - d]$

such that



Then by the cut criterion, $\tilde{F}$ is more variable than F



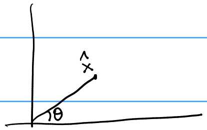
In this case, as variance increases, then so does the time constant. Is this true in general?

Alm and Deijfen propose a new quantity:

$E\left[\min\{t_1, t_2, t_3, t_4\}\right] =: E_4[t]$

that appears to "determine" μ_F.

This is how they setup their simulations:



$\hat{x}$ unit vector wrt Euclidean norm in direction θ.

They choose $\theta = 0, 14, 27, 37, 45^\circ$ and find μ by growing the cluster until it hits a line.

They pick several different distributions:

1) Unif $(0,1)$

2) $Exp(1)$

$\boxed{5)\ U(0.1, 0.9)}$

Weird outlier

3) $\Gamma(2,2)$ Gamma distribution

4) $\Gamma(3,3)$, $\Gamma(4,4)$

5) $\hat{F}i(k)$ $k = 1,2,3,4$

$\text{Tail}(x) = \frac{1}{(1+x)^4}$

(Previously Eden, Richardson had focused only on the Exponential case.)

I'll return to this paper many times. For example, it has

1) Non random fluctuations (Fig 2) Skip

2) A linear graph relating $\mu(e_1) = 1.5\, E_4[\tau]$ and $\mu(\hat{x}_{45^\circ}) = 1.56\, E_4[\tau]$ (Fig 4 and 5 on page 13)

3) 2 distributions $F, \tilde{F}$ with the same mean and $Var(F) = Var(\tilde{F})$ but $\mu_{\tilde{F}} < \mu_F$

"Proving" that variance alone cannot successfully describe the time constant (Fig 3)

4) An observation that $\mu(\hat{x}_\theta)$ is increasing on the interval $\theta \in [0, \frac{\pi}{4}]$ (It's enough to describe μ on an octant and use symmetry)

Research Problem: Prove this. I think this might be quite hard to do, BUT I think a more sensible norm here is the $|x|_1$ norm. I wonder if the monotonicity is still observed in this norm.

5) They also observe that the "most heavy tailed" distributions are the most circle like (Fischer distributions on page 16, Fig 9)

6) In Fig 10 they simulate a bunch of $\mathrm{Exp}(1) + c$ distributions and show that the shape gets more and more like a diamond.

Ex: Prove this by considering $\frac{\mathrm{Exp}(1)+c}{c}$ and applying Cox - Kesten.

7) Then they show that $\chi = 0.322\ldots$ $\xi = 0.667$ in their experiments (Fig 16)

8) Alm and Deijfen compute $T(0, L_n)$ instead of $T(0, ne_1)$ (this works for time constant computation due to a theorem of Wierman and Reh). The conjecture that

$$\mathrm{Var}(T(0, L_n)) \leq \mathrm{Var}(T(0, ne_1)).$$

Can one prove this?

8) Finally, let $E_4(\tau^2) := E\left[\min_{i=1,\dots,4} \tau_i^2\right]$

In $\hat{F}_i(0.3)$ with tail $\frac{1}{(1+x)^\gamma}$ we have $E_4[\tau] < \infty$

but $E_4(\tau^2) = +\infty$.

In this case, Richardson's theorem is not true.

$$\mathbb{P}\left((1-\varepsilon)\, \mathcal{B} \subset \frac{\mathcal{B}_n}{n} \subset (1+\varepsilon)\, \mathcal{B} \right) \not\to 1$$

This is seen in Fig 15. The boundary of the infected set is shown in red. The red regions in the interior enclose regions that have not yet been infected. This is the content of the next lecture.

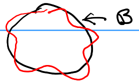
The Cox - Durrett limit shape theorem.

Recall p_c, the threshold for Bernoulli bond percolation.

Suppose
$$\begin{cases} \mathbb{E}\left[\min_{i=1,\dots,2d}\{t_i\}^d\right] < \infty & \left(\begin{array}{c}\text{a bit more than}\\ \text{time constant condition}\end{array}\right) - \#1 \\ F(0) < p_c & \left(\begin{array}{c}\text{no percolation of}\\ 0\text{s} \Rightarrow \mu(e_1) \neq 0\end{array}\right) - \#2 \end{cases}$$

GOOD measures

<u>Theorem</u>: (Cox and Durrett)



Let $\mathcal{B} = \{x \mid \mu(x) \leq 1\}$

Let $B(n) = \{x \in \mathbb{R}^d \mid T(0,x) \leq n\}$

Then for any $\epsilon > 0$
$$P\left((1-\epsilon)\mathcal{B} \subset \frac{B(n)}{n} \subset (1+\epsilon)\mathcal{B} \text{ for all large } n\right) = 1$$

Ahn and Deijfen also have many intriguing questions: Is $\mu \geq E_4[\tau]$ in general?

This would be LOVELY to prove.