Lec 17 Non random fluctuations

Since $\lim_{|x|_1 \to \infty} \frac{T(0,x)}{|x|_1} = g(z)$ where $\frac{x}{|x|_1} \to z$

Fluctuations $= o(|x|_1) = T(0,x) - \mathbb{E}[T(0,x)]$

$+ \underbrace{\mathbb{E}[T(0,x)] - g(x)}$

deterministic or nonrandom

The first term is controlled by the variance.

$$\sqrt{\mathbb{E}[(T(0,x) - \mathbb{E}T(0,x))^2]} = \sqrt{\mathrm{Var}(T(0,x))} \approx |x|^{\chi}$$

$\chi = \frac{1}{3}$ (Conjecturally). But the best bound we have is $\mathrm{Var}(T(0,x)) \lesssim \frac{|x|}{1+\log|x|} \Rightarrow \chi \leq \frac{1}{2}$

It's a major open problem to improve this bound from

$1/2$ (diffusive or Gaussian behavior) to

$1/3$ (Random-matrix subdiffusive behavior)

Let $\boxed{h(x) = T(0,x)}$ for this part of the lecture (Alexander's notation)

We wish to control

$$\mathbb{E}[T(0,x)] - g(x) = h(x) - g(x).$$

Exercise: Show $h(x) \geq g(x) \quad \forall x \in \mathbb{Z}^d$

Kesten (1993)

$$g(x) \leq h(x) \leq g(x) + C|x|^{7/8}(\log|x|)^{1/d+2}$$

This is a pretty bad bound and one really ought to improve the nonrandom fluctuations exponent (call it γ) to $\gamma \approx \frac{1}{3}$

We dont know this,

1) but Alexander claims Kesten 1993 provides numerical evidence

2) In last passage percolation with exponential wts its can be proved.

Kesten's bound is not great, but can we at least improve it to $\gamma \leq \frac{1}{2}$ like $\chi \leq \frac{1}{2}$?

Alexander's brilliant argument shows the following:

average passage time
time constant.

Thm: (Alexander 93/97):

$$h(x) - g(x) = \mathbb{E}\,T(0,x) - g(x) \leq C|x|^{1/2} \log|x| \quad \text{(\#1)}$$

$\boxed{GAP(\nu,\varphi)}$: h satisfies $GAP(\nu,\varphi)$ if (#1) holds $\forall x$.

we will show that $\nu = \frac{1}{2}$, $\varphi = \log|x|$ for FPP

Ingredients:

1) Alexander's brilliant CHAP / GAP exponent improvement iteration.

2) CHAP (Convex Hull Approximation Property) for FPP

BK inequality

$P(A \circ B) \leq P(A)P(B)$

Gaussian concentration

$$P\left(T(0,x) - \mathbb{E}T(0,x) > t\sqrt{|x|}\right) \leq C_1 e^{-t^2/2C_2}$$

$$\mathbb{E}\,T(0,X) - g(X) \geq 0$$

deterministic

$$\leq |x|^{\nu} \varphi(|x|)$$

$$\sqrt{\operatorname{Var}(T(0,x))} \lesssim \frac{|x|^{1/2}}{\sqrt{\log|x|}}$$

$\nu = \frac{1}{3}$. $\nu = \frac{1}{2}$

1) BK ineq.

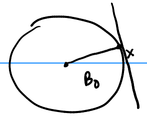
2) $P\left(T(0,x) - ET(0,x) > t\sqrt{|x|}\right) \lesssim e^{-Ct^2}$

Recall $g(x) = \lim_{n\to\infty} \frac{T(0,nx)}{n}$, $g(x)$ is a norm

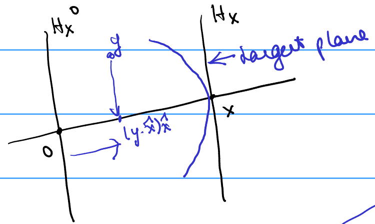
$$C_1 |x|_\infty \leq g(x) \leq C_2 |x|_1$$

B_0 = limit shape $= \{x : g(x) \leq 1\}$

H_x = hyperplane tangent to ∂B_0 at x



$H_x^{\circ} = H_x$ translated to the origin.



$\{y : g(y) = g(x)\}$

$g(\lambda x) = \lambda g(x)$

homogeneity

linear approximation to time constant.

Define $\quad g_x(y) = g\left(\left(y \cdot \frac{x}{|x|}\right) \cdot \frac{x}{|x|} \right)$

That is $g_x(y)$ is the unique linear functional

that is 1) $g_x(y) = 0$ on H_x°

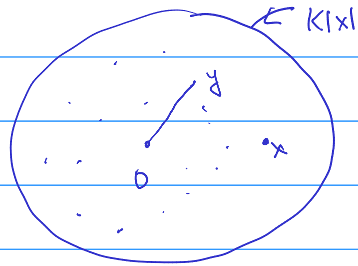
2) $g_x(x) = g(x)$

POLL.

Would you be willing to meet on Friday night

YES | NO

Exercise: Show $|g_x(y)| \leq g(x)$ using the symmetries of g. and convexity.



$$Q_x\,\overline{(\nu,\varphi)} = \{ y \in \mathbb{Z}^d : |y| \leq K|x|,\ g_x(y) \leq g(x),$$
$$h(y) \leq g_x(y) + C\,|x|^{\nu}\varphi(|x|) \}$$

GAP

So K and C are constants, but we will suppress them

$\varphi : [1,\infty) \to \mathbb{R}$ is a nonnegative increasing fn.

$h(y) = \mathbb{E}\,T(0,y)$

$h(y) - g_x(y) \leq C|x|^{\nu}\varphi(|x|)$

$\nu < 1$

Q_x is the set of good increments y (good for $h(y)$)

That help it make good progress towards $g(x)$

1) $g_x(y) \le g(x)$ makes sure we're making "progress towards $g(x)$"

2) $h(y) \le \underbrace{g_x(y)}_{\le g(x)} + C|x|^\nu \log|x|$ makes sure the increments are GOOD.

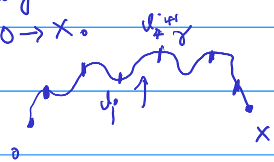
error on these increments is small and there are not too many increments.

$\Uparrow$

$v_{i+1} - v_i \in$ "good increments" $\in Q_x$

Idea: How would one try to prove gap?

Follows from considerations of the passage time on a geodesic from $0 \to x$.



Suppose $\exists$ m good increments in $Q_x(\nu, \varphi)$

st $\quad x = \sum_{i=1}^{m} y_i$

$$\text{Then} \quad h(x) \underset{\text{subadditive}}{\le} \sum_{i=1}^{m} h(y_i) \underset{\text{GAP}}{\le} \sum_{i=1}^{m} \overbrace{g_x}^{\text{linear}}(y_i) + \underbrace{C|y_i|^\nu \varphi(|y_i|)}_{\text{error by GAP}}$$

$$\le g_x(x) + C K^\nu m |x|^\nu \varphi(K|x|)$$

$|y_i| \le K|x|$

1) $\exists$ m good increments in each path

2) m is not too large.

So if we can show that m is not too large, then we're in business. BUT THIS IS HARD TO SHOW

(Alexander says that this does not seem natural)

GAP $\equiv$ General Approximation property

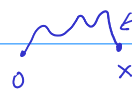
What do we replace this by? Property called CHAP

CHAP $\equiv$ Convex Hull Approximation Property.

Lemma: Suppose for each $x \in \mathbb{Q}^d$ st $|x| \ge M$ $\exists$ $n \ge 1$

and γ from 0 to nx with a sequence $0 = v_0, v_1, \dots, v_m = nx$

st $m \le a \cdot n$ (fixed constant a), $v_i - v_{i-1} \in Q_x$ for all $1 \le i \le m$



cannot show that is made up of good increments.

Then h satisfies something called CHAP (Convex Hull Approximation Property)

The difference is that here we have $\exists$ n st...

(One would expect to see some sort of Borel - Cantelli to say ALL n cannot be bad)

Consider the increments $\{v_i - v_{i-1}\}$. They may repeat, so let $\beta_n(y,\gamma)$ be the # of times $v_i - v_{i-1} = y$

Then write $nx = \sum_{y \in Q_x} \beta_n(y,\gamma) y$ ← (#0)

can sum over all Q_x by allowing $\beta_n = 0$

By the assumption $\sum \beta_n(y,\gamma) \leq an$ $(m = \#$ of increments$)$

by assumption in the Lemma

let $\alpha = \sum \frac{\beta_n(y,\gamma)}{n}$ $\quad a(y,\gamma)$

and $\frac{x}{\alpha} = \sum_{y \in Q_x} \frac{\beta_n(y,\gamma)}{m\alpha} y$ $\quad$ Convex combination in Q_x

Now these coefficients sum to 1.

and thus $\frac{x}{\alpha} \in$ Convex Hull (Q_x) → CHAP

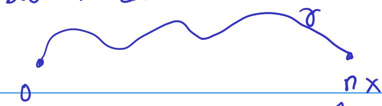
and $g_x(nx) = n g_x(x) = \sum \beta_n(y,\gamma) g_x(y)$

$n g(x) \leq g(x) \sum_{y \in Q} \beta_n(y,\gamma) = g(x) n \alpha$

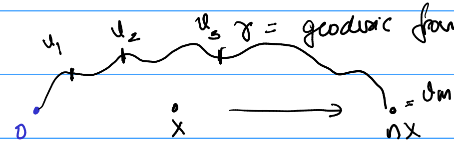
$\Rightarrow \alpha \geq 1$.

$g_x(y) = g\left(\left(y \cdot \frac{x}{|x|}\right) \frac{x}{|x|}\right)$

But I can show that for some n



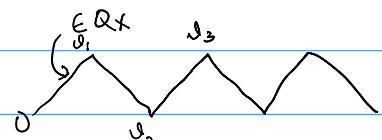
rescale then I can show $\exists$ $m \leq an$ many good increments.



$nx = \sum_{y \in Q_x} \beta_n(n,\gamma) y$

↳ fixed set of points in $\mathbb{Z}^d$

Fix a seq $v_1 \cdots v_m$. Then you see



$v_1 - 0 = y$ $\quad \beta_n(y,\gamma) = 3$

$v_3 - v_2 = y$

If we can show that $\exists$ a bunch of good increments on γ from 0 to nx then we have CHAP.

GAP: $h(x) - g(x) \le |x|^{\nu} \varphi(|x|)$

$h(x) = \mathbb{E}\,T(0,x)$

CHAP: h satisfies $CHAP(\nu, \varphi)$ if $\exists\, a$ st for $a \geq \alpha \geq 1$

$\frac{x}{\alpha} \in Co(Q_x(\nu,\varphi))$ $\quad \forall x \in \mathbb{Q}^d,\ |x| \geq M$ ← ignore the constants.

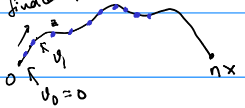
GOOD INCREMENTS.

Exercise: In d dimensions if $z \in Co(A)$, then $\exists$ (any set)

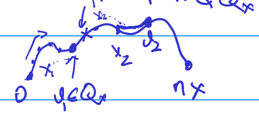
$\{y_i\}_{i=1}^{d+1}$ st $z = \sum_{i=1}^{d+1} \alpha_i y_i$ with $1 \geq \alpha_i \geq 0$ $\sum_{i=1}^{d+1} \alpha_i = 1$

any point in the convex hull can be written as a convex combination of $d+1$ increments.

How to construct good increments:



So you're trying to find good increments that are as far apart as possible



Find a Q_x skeleton. deterministic set

$x_2 - x_1$'s in Q_x

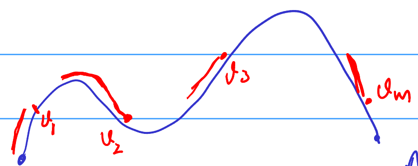
↳ $v_2 - v_1$ is in Q_x as well.

Theorem 1.8: Suppose h is subadditive and $h(x) \leq c|x|$ $\quad h(x) = \mathbb{E}\,T(0,x)$

POLL:

For our $h(x) = \mathbb{E}\,T(0,x)$, a good constant c is:

A	B	C	D
1	$\mathbb{E}[\tau_e^2]$	$\mathbb{E}[\tau_e]$	$\mathbb{E}[e^{\lambda \tau_e}]$



Annals of Prob. ↳ Invited Paper?

$\varphi(x) = x$

$(\mathbb{E}[\tau_e])^2 \leq \mathbb{E}[\tau_e^2]$

If h satisfies $GAP(1,\varphi)$ and

If h satisfies $CHAP(\nu, \varphi)$ THEN h satisfies $GAP(\nu, \varphi)$

improve this exponent

This is Alexander's main theorem.

GAP is what we wanted to prove.

$h(x) \le C|x|$ $\quad g(x) = \lim_{n\to\infty} \frac{h(nx)}{n} \le C|x|$

$h(x) - g(x) \ge 0$

$\Rightarrow h(x) - g(x) \le C|x|$

h already satisfies $GAP(1, \varphi)$

Note: sublinear $h \Rightarrow h(x) \le C|x| \Rightarrow$ It satisfies $GAP(1, \varphi)$ ← exponent 1.

It relies on iterating the following result.

Prop: Suppose $h \le C|x|$, h satisfies both $GAP(\beta, \varphi)$ and $CHAP(\nu, \varphi)$ then h satisfies $GAP(\beta', \varphi)$ where

$$\beta' = \frac{\beta}{1+\beta-\nu} < \beta$$

fixed point of this iteration is:

$\beta = f(\beta)$ $\quad \cancel{\beta} = \frac{\cancel{\beta}}{1+\beta-\nu}$

$1+\beta-\nu = 1$

What is the fixed point of this iteration?

POLL

A	B	C
β	ν	$1/2$

Pf: Start with x, and pick a parameter q. a free parameter

$1 << q << |x|$ and we will optimize over q. Assume $|x|$ is large, and q is st

by CHAP, $\frac{x}{q} \in Co(Q_x)$

$$\frac{x}{q} = \sum_{i=1}^{d+1} \alpha_{q_i} y_{q_i} \quad \text{and of course } 1 \le \sum_{i=1}^{d+1} \alpha_{q_i} \le a$$

is the same a in the definition of CHAP

We assume $GAP(\beta, \varphi)$

Assume $CHAP(\nu, \varphi) \Rightarrow GAP(\beta', \varphi)$

These α_{q_i} are the $\beta_n(y)$ we considered

$\mathbb{Z}^d \quad \mathbb{Z}^d$

Thus $x = \underbrace{x^*} + x - x^*$

↳ Essentially $x/q * q \to$ CHAP control the error

where $x^* = \sum \alpha_i y_{q_i}$ ← grid increments.

We can use subadditivity to bound $\overline{h(x^*) - g_x(x^*)}$ ← Error 1

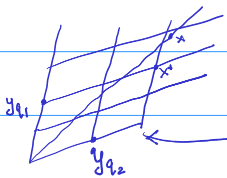
and then $\overline{GAP(\beta, \varphi)}$ to bound $h(x-x^*) - g_x(x-x^*)$ ← Error 2

Then we optimize over q to improve the error!

THIS IS THE MAIN IDEA OF THE ITERATION:

break up the error and control part of it with CHAP

and the rest with GAP. This improved the EXPONENT!



$d=2 \quad Q_x = \{y_{q_1}, y_{q_2}, \vec{0}\}$

take integer combinations to form a lattice with Q_x.

$$\frac{x}{q} = \sum_{i=1}^{d+1} \alpha_{q_i} y_{q_i} \quad \text{with } \alpha_{q_i} \geq 0 \qquad \frac{x}{\alpha q} \in Co(Q_x)$$

and $\alpha \leftarrow \sum_{i=1}^{d+1} \alpha_{q_i} \in [1, \dot{a}]$

α_{q_i} may not be integers. I want to pick integer combination of the y_{q_i}

Then set $x^* = \sum_{i=1}^{d} \lfloor q \alpha_{q_i} \rfloor y_{q_i}$ ← floor → multiply by q and get integer part.

$$h := ET(0, x) : \mathbb{Z}^d \to \mathbb{R}$$

and $x - x^* = \sum_{i=1}^{d+1} \gamma_{q_i} y_{q_i}$

$$\gamma_{q_i} = \underbrace{q\alpha_{q_i} - \lfloor q\alpha_{q_i} \rfloor}_{\text{integer approximation error.}} \in [0,1)$$

subadditivity

$$h(x^*) \leq \sum_{i=1}^{d+1} \lfloor q\, \alpha_{q_i} \rfloor\, h(y_{q_i})$$

goodness of increment ($\mathbb{Q}x$) their errors are small

$$\leq \sum_{i=1}^{d+1} \lfloor q\, \alpha_{q_i} \rfloor \left[g_x(y_{q_i}) + C \left|\frac{x}{q}\right|^{\nu} \varphi\left(\left|\frac{x}{q}\right|\right)\right]$$

linearity

$$\leq g_x(x^*) + C\, q^{1-\nu} |x|^{\nu} \varphi(|x|) \qquad (\#2)$$

free parameter

monotonicity and $q > 1$

To bound $h(x - x^*)$ assume first that $|x - x^*| \geq M$ and

$$g(y_{q_i}) - g_x(y_{q_i}) \leq h(y_{q_i}) - g_x(y_{q_i}) \leq C \left|\frac{x}{q}\right|^{\nu} \varphi\left(\frac{|x|}{q}\right)$$

$$\Rightarrow \sum_{i=1}^{d+1} \gamma_{q_i} \left[g(y_{q_i}) - g_x(y_{q_i}) \right] \leq C\,(d+1) \left|\frac{x}{q}\right|^{\nu} \varphi(|x|)$$

Since $x - x^* = \sum \gamma_{q_i} y_{q_i}$

$$|x - x^*| \leq \sum_{i=1}^{d+1} |y_{q_i}| \leq C\,(d+1)\, \frac{|x|}{q} \qquad (\#2a)$$

Thus

$$h(x - x^*) \leq g(x - x^*) + C\,|x - x^*|^{\beta} \varphi(|x - x^*|)$$

GAP

subadditivity

$$\leq g_x(x - x^*) + \sum_{i=1}^{d+1} \gamma_{q_i} \left(g(y_{q_i}) - g_x(y_{q_i}) \right)$$

add and subtract

Assumption of GAP(β, φ)

$$+ C \frac{|x|^{\beta}}{q^{\beta}} \varphi\left(C \frac{|x|}{q} \right) \qquad (\#3)$$

(#2a)

Combining (#2) and (#3)

$$h(x) \leq h(x^*) + h(x-x^*)$$

$$\leq g_x(x) + C_1 q^{1-\nu} |x|^{\nu} \varphi(|x|)$$

$$+ g_x(x-x^*) + C_2 \frac{|x|}{q^{-\nu}} \varphi(|x|) + C_3 \frac{|x|^{\beta}}{q^{\beta}} \varphi(|x|)$$

$g_x(x) = g(x)$

$$\leq g(x) + \Big(\underbrace{C_1 q^{1-\nu} |x|^{\nu}}_{\text{Error 1}} + \underbrace{C_2 |x|^{\beta} q^{-\beta}}_{\text{Error 2}}\Big) \varphi(|x|)$$

So simply choose q st

$$q^{1-\nu}|x|^{\nu} = |x|^{\beta} q^{-\beta} \Rightarrow q^{1-\nu+\beta} = |x|^{\beta-\nu}$$

$$\Rightarrow q = |x|^{\frac{\beta-\nu}{1+\beta-\nu}}$$

$$\Rightarrow |x|^{\beta} q^{-\beta} = |x|^{\beta - \frac{\beta(\beta-\nu)}{1+\beta-\nu}} = |x|^{\frac{\beta}{1+\beta-\nu}}$$

$\rightarrow$ A better error! $\quad h(x) - g(x) \leq C|x|^{\frac{\beta}{1+\beta-\nu}}$

Amazing, right?

Lesson: Strange scaling, relaxing and playing terms off against each other improves inequalities.

Lesson: Terry Tao talks about this in a blog post about Cauchy-Schwarts/Hölder.

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

introduce parameter λ $\quad a, b > 0 \quad$ Young's inequality.

$$\lambda a \frac{1}{\lambda} b \leq \frac{\lambda^p a^p}{p} + \frac{b^q}{\lambda^q q}$$

New parameter λ

Integrate to get

$$\int ab \le \frac{\lambda^p}{p} |a|_p^p + \frac{1}{\lambda^q} \frac{|b|_q^q}{q}$$

$\frac{|a|_p |b|_q}{p}$

Set $\lambda^p |a|_p^p = \frac{1}{\lambda^q} |b|_q^q$

$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p+q = pq$

$$\Rightarrow \lambda^{p+q} = \frac{|b|_q^q}{|a|_p^p}$$

$$\Rightarrow \lambda^p = \frac{|b|_q}{|a|_p^{p/q}} \left(\frac{p}{q}\right)^{1/q}$$

$$\Rightarrow \frac{\lambda^p |a|_p^p}{p} = \frac{|b|_q |a|_p}{p}$$

$p\left(1 - \frac{1}{q}\right) = 1$

$$\Rightarrow \boxed{\int ab \le |a|_p |b|_q}$$

Extremely powerful idea in inequalities.

You introduced a parameter λ and optimized over it.

There's a bunch of technical stuff to ensure that constants don't muck things up terribly under iteration.