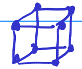
lec16

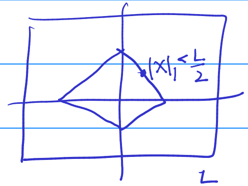
Fourier-Walsh Expansions (discovering a useful orthonormal basis on the hypercube $L^2(\{-1,1\}^n)$ with uniform measure]

$\hat{\Omega} = \{1,2\}^n$, edge wts in a box of length $2L+1$ centered at the origin

$\Omega = \{-1,1\}^n$ to be the edge weights



If $2|x|_1 < L$ Then the geodesic must stay inside $[-L,L]^d$.



If $|\Gamma| > L$ then $T(\Gamma) > L$

$T(0,x) \leq 2|x|_1 < T(\Gamma)$

Thus $T(0,x) : \hat{\Omega} = \{1,2\}^n \longrightarrow \mathbb{R}$ (we may restrict to a bounded box)

$e(1), e(2), \dots e(n)$ that's a list of all edges.

Now in the Fourier walsh setup, we have $\Omega = \{-1,1\}^n$ $\mathbb{P}(\omega_i = \pm 1) = \frac{1}{2}$

let $\boxed{\hat{\omega}_i = \frac{1}{2}\omega_i + \frac{3}{2}}$ just a shift so I can do harmonic analysis on $\{-1,1\}^n$ but still do FPP with positive wts.

Then $T(0,x)(\hat{\omega}) = T(0,x)\left(\frac{1}{2}\omega + \frac{3}{2}\right)$ $\omega \in \{-1,1\}^n$

shifted weights living in $\{1,2\}^n$

Now, consider the following basis on $L^2(\Omega, \mathbb{P})$ uniform meas on Ω

This is a 2^n dim. vector space (2^n points in Ω)

For each $S \subset \{1, 2, \dots, n\}$ (coordinates)

Let $\underbrace{\chi_S(\omega)}$ = $\prod_{i \in S} \omega_i$ $\quad \omega \in \Omega$

χ_S is constant on the cylinder $C(\omega, S)$ (in the BK notation)

$\chi_S(\omega) \in \{-1, 1\}$.

This fn only cares about coordinates in S

$C(\omega, S) = \{\omega' \in \Omega : \omega'_i = \omega_i \ \forall i \in S\}$

$\uparrow$ cylinder set.

POLL: What is the dimension of $L^2(\Omega)$?

$f : \{-1, 1\}^n \longrightarrow \mathbb{R}$

A	B	C
$\log n$	n	2^n

There are 2^n elements $\in \{-1,1\}^n$. $\{\mathbb{1}_{\vec{\omega}_i}\}_{i=1}^{2^n}$ are the indicators of the

$f(\omega) = \sum_{i=1}^{2^n} a_i \mathbb{1}_{\vec{\omega}_i}(\omega)$ is an expansion.

These are linearly independent. Thus $\dim(L^2(\Omega)) = 2^n$

There are 2^n elements of the form $\{\chi_S(\omega)\}_{S \subset \{1, \dots, n\}}$.

$$\langle f, g\rangle = \mathbb{E}[fg] = \sum_{\omega\in\Omega} \frac{1}{2^n} f(\omega)g(\omega)$$

Suppose $S \neq T$

$$\langle \chi_S, \chi_T\rangle := \mathbb{E}[\chi_S \chi_T] \rightarrow \mathbb{E}\left[\prod_{i\in S} \omega_i \prod_{j\in T}\omega_j\right]$$

$$= \mathbb{E}\left[\left(\prod_{i\in S\setminus T}\omega_i\right)\left(\prod_{j\in S\cap T}\omega_j^2\right)\left(\prod_{k\in T\setminus S}\omega_k\right)\right] \rightarrow \prod_{i\in S\setminus T}\mathbb{E}[\omega_i] \prod_{i\in S\cap T} \mathbb{E}[\omega_i^2] \prod_{i\in T\setminus S}\mathbb{E}[\omega_i^2] \to 0$$

is nonempty

Note $\mathbb{E}[\omega_i] = 1\cdot\frac{1}{2} - 1\cdot\frac{1}{2} = 0 \quad \mathbb{E}[\omega_i^2] = 1^2\cdot\frac{1}{2} + 1^2\cdot\frac{1}{2} = 1$

$= 0$

$$\Rightarrow (\chi_S, \chi_T) = \begin{cases} 1 & \text{if } S = T \\ 0 & S\neq T\end{cases}$$

So the $\{\chi_S\}_S$ are 2^n orthogonal vectors in a 2^n dim space, so it must be a basis. In fact they are orthonormal.

walsh coefficients.

Thus we can write $f(\omega) = \sum_S \hat{f}_S \chi_S(\omega)$

$$(f, \chi_T) = \mathbb{E}\left[\sum_S \hat{f}_S \chi_S(\omega)\chi_T(\omega)\right] = \mathbb{E}\sum_S \hat{f}_S \chi_S(\omega)\chi_T(\omega)$$

$$= \hat{f}_T \qquad = \sum \hat{f}_S\, \mathbb{E}[\chi_S\chi_T] = \hat{f}_T$$

POLL What is $\langle f, \chi_\phi\rangle$? $\mathbb{E}[f1]$ $\quad \chi_\phi$? $\quad \phi \subset \{1,\dots,n\}$

A	B	C
0	1	$\mathbb{E}[f]$

$\chi_S = \prod_{i\in S}\omega_i$

What should χ_ϕ be?

$\mathbb{E}[\chi_\phi^2] = 1 \quad \mathbb{E}[\chi_\phi \chi_S] = 0$ $\quad S\neq\phi$

$\chi_\phi = 1$ identically.

Parseval's Formula:

$$\mathbb{E}f^2 = \mathbb{E}\left[\left\{\sum_S \hat{f}_S\chi_S(\omega)\right\}\left\{\sum_T \hat{f}_T\chi_T(\omega)\right\}\right]$$

$$= \sum_S\sum_T \mathbb{E}[\chi_S\chi_T]\hat{f}_S\hat{f}_T = \sum_S \hat{f}_S^2 = \|f\|_2^2$$

$S=T$

nonempty S

$$= \sum_S \hat{f}_S^{\,2} = \overbrace{\sum_{S \neq \phi} \hat{f}_S^{\,2}}^{\text{nonempty } S} + \underbrace{\hat{f}_\phi^{\,2}}_{}\quad \hat{f}_\phi^{\,2} = (\mathbb{E}[f])^2$$

$$\mathbb{E}[f^2] = \sum_{S \neq \phi} \hat{f}_S^{\,2} + \mathbb{E}[f]^2$$

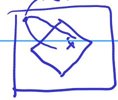
$$\mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2$$

$$= \sum_{S \neq \phi} \hat{f}_S^{\,2}$$

We will provide a lower bound for the variance.

Look at nice subsets S, where $\hat{f}_S$ is easy to compute and dropthrough of the hemisso

edge



$S = \{k\}$

f will independs of χ_S

$\mathbb{E}[f\chi_S] = \mathbb{E}[f]\,\mathbb{E}[\cancel{\chi_S}]^{\,0}$

Recall $\hat{\omega} = \frac{1}{2}\omega + \frac{3}{2} = g(\omega)$, and thus $\hat{\omega} \in \{1, 2\}$

Let $f = T(0, x, \hat{\omega}) = T(0, x, g(\omega))$ = passage time

I want to write $T(0, x, g(\omega)) = f(\omega)$ in the Fourier-Walsh basis.

Find the coefficients: $\hat{f}_S = \mathbb{E}[f\chi_S]$

Then, $\mathrm{Var}(f) = \sum_S \hat{f}_S^{\,2}$

We will focus on coefficients of the form $S = \{i\}$

for $i = 1, \dots, n$.

Then $\hat{f}_{\{i\}} = \mathbb{E}[f(\omega)\,\omega_i]$ $\quad \omega_i = \chi_{\{i\}}$

Let's write $\omega = (\omega_i, \omega_1, \omega_2, \dots \omega_n)$ (does not have i)

$= (\omega_i, \omega_{i^c})$ $\qquad f(\omega) = f(\omega_i, \omega_{i^c})$

where $\omega_{i^c} = \{\omega_k\}_{k \neq i}$ $\quad$ Then

$$= \mathbb{E}\left[f(1, \omega_{i^c})\, 1 \cdot \underbrace{\tfrac{1}{2}}_{P(\omega_i = 1)} - f(-1, \omega_{i^c})\, \tfrac{1}{2}\right]$$

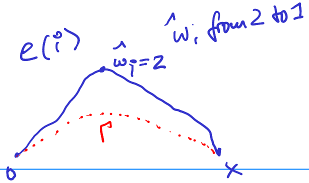
$$= \frac{1}{2}\, \mathbb{E}\left[f(1, \omega_{i^c}) - f(-1, \omega_{i^c})\right]$$

"the effect of the i^{th} coordinate on the passage time"

Influence of the i^{th} coordinate.

For $T(0, x, \hat{\omega})$, $\hat{\omega} \in \{1,2\}$, this becomes

$= T$



$$= \frac{1}{2}\,\mathbb{E}\left[T(2,\hat\omega_i c) - T(1,\hat\omega_i c)\right]$$

$$= \mathbb{E}\Big[\underbrace{\big(T(2,\hat\omega_i c) - T(1,\hat\omega_i c)\big)}_{\text{that quantity does not depend on } \hat\omega_i}\mathbb{1}_{\{\hat\omega_i = 2\}}\Big] = \mathbb{E}\left[T(2,\hat\omega_i c) - T(1,\hat\omega_i c)\right]\overbrace{\mathbb{E}\left[\mathbb{1}_{\hat\omega_i = 2}\right]}^{1/2}$$

$$= \mathbb{E}\Big[\big(T(2,\hat\omega_i c) - T(1,\hat\omega_i c)\big)\mathbb{1}_{\{\hat\omega_i = 2\}}\Big] \quad \text{— (\#1)}$$

$T(2,\hat\omega_i c) - T(1,\hat\omega_i c))\mathbb{1}_{\{\hat\omega_i = 2\}} = 1$ if $e(i) \in \overline{GEO}(0,x)$

Quantity is non zero only if $\overline{GEO}(0,x) \ni e(i)$

Since if $\overline{GEO}(0,x)$ does not contain $e(i)$ when $\hat\omega_i = 2$

$$T(\Gamma,(2,\hat\omega_i c)) = T(\Gamma,(1,\hat\omega_i c)) \Rightarrow T(2,\hat\omega_i c) = T(1,\hat\omega_i c)$$

In this case, if $GEO(0,x) \ni e(i)$ and $\mathbb{1}_{\{\omega_i = 2\}}$

Then $T(2,\hat\omega_i c) - T(1,\hat\omega_i c) = 1$

$$(\#1) = \mathbb{P}\left(e(i) \in \overline{GEO}(0,x),\ \hat\omega_i = 2\right)$$

Thus

$\mathbb{E}[f\,\chi_{\{i\}}]^2$

$$\mathrm{Var}(T(0,x)) \geq \sum_{i} \mathbb{P}\left(e(i) \in GEO(0,x),\ \hat\omega_i = 2\right)^2$$

i ← all coordinates

$$= \sum_i P(F_i)^2$$

This was the form of the Piza newman bound.

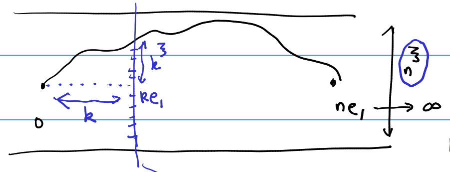
How to evaluate this sum, Licea - Newman 1990s.

Newman

Newman - Piza } PhD theses?

Newman - Licea }

Heuristics : Suppose



(with high probability)

In d dimensions

Better still, assume that at each hyper plane $\{x\cdot e_1 = k\} \eqqcolon H_k$

Each edge $y \in H_k$ is equally likely to be in $GEO(0,x)$

as long as they are within distance k^ξ : $|y - ke_1| < k^\xi$

$\sum_{e(i)} P(F_i)^2$ (How big can this sum be)

Then $\sum_{e(i)} P(e(i) \in GEO(0,x), \tilde{\omega}_i = 2)^2$ removed this condition

$$\leq \sum_{k=1}^{n} \sum_{e(i)\in H_k} P(e(i) \in GEO(0,x))^2$$

$$= C \sum_{k=1}^{n} \underbrace{(k^\xi)^{d-1}}_{\#\text{ of edges in } H_k \cap \{\text{tube of radius } k^\xi\}} \left(\frac{1}{(k^\xi)^{(d-1)}}\right)^2 \approx \int_1^n \frac{1}{x^{\xi(d-1)}}\,dx \approx C\, n^{-\xi(d-1)+1}$$

$$\approx x^{1-\xi(d-1)}$$

We know $|\xi| \leq 1$ when $F(0) < p_c$ (why?)

So $m = 1 - \xi(d-1) \geq 2 - d$

This is not so great for all d, but especially so for $d > 2$.

If $\xi = \frac{2}{3}$ (Conjecture) in $d = 2$ $\quad 1 - \xi(2-1) = \frac{1}{3} < $ "Truth" $= \frac{2}{3}$

$\mathrm{Var}(T(0, ne_1)) \approx n^{2/3}$ in $d=2$

ξ was supposed to be the transversal fluctuations exponent.

Anirudh : $2\xi - 1 = \chi$ under strong existence assumptions for ξ and χ (Chatterjee 2012-2013 Annals of Math)

$\xi < 1$

$\mathrm{Var}(T(0,x)) \geq \sum P(F_i)^2$

how good is this bound in the best case scenario.

$\mathrm{Var}(T(0,x)) \geq C\, n^{-\xi(d-1)+1}$

$1 - \xi(d-1) > 0$

Even if you assume lots of things about the problem, the singleton Fourier coefficient do not give you the right order for the variance.

But $\chi = 1/3$, or $\mathrm{Var}(T(0,x)) \asymp |x|_1^{2/3}$ (expected)

So this is not quite the right order.

Is there any way to improve this bound? If $\xi < 1$ then $1 - \xi > 0$. THIS WOULD BE A HUGE contribution. Could one show this outside the percolation cone?

Prove this bound for $d \geq 3$

$\log |x|_1$ is only known for $d=2$

A Talk about 1) Conc. of meas style inequalities – Jiaming, Hypercontractivity and log Sobolev ineq. (Analytic)

B 2) Alexander's result (Hard, but extremely interesting paper)

↳ BK inequality, Kesten's inequality.

attack the so called fluctuations exponent

↳ nonrandom

$$\underbrace{|\mathbb{E}T(0,x) - g(x)|} + \mathbb{E}\big[|T(0,x) - \mathbb{E}T(0,x)|\big]$$

$$\gtrsim \sqrt{\mathrm{Var}(T(0,x))} \gtrsim |x|^{1/3}$$

C 3) Geodesic behavior and Busemann fns.

↳ Algorithms

Talk a little bit some ergodic theoretic