lec 15 (Piza - Newman, 90s)

**Theorem:** let $d = 2$ and let $I = \inf_{x} \{ x : F(x) > 0 \}$

Assume $\mathbb{E}[\tau_e^2] < \infty$, $Var(\tau_e) > 0$.

doesn't percolate ← $I = 0$ $F(0) < p_c$ $\qquad \vec{p}_c > p_c$

$I > 0$ $F(I) < \vec{p}_c$ $\qquad$ Then $\exists\ B > 0$.

$$Var(T(0,x)) \geq B \log|x|, \quad \forall x \in \mathbb{Z}^2$$

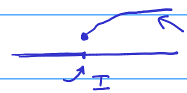
rely on dropping a bunch of Fourier coeff.

1) Kesten's lemma: Paths of length n cannot have very small passage times

inf support of the random edge weights.

$Var(\tau_e) = 0 \Rightarrow \tau_e = c$ almost surely

Ex.




Relies on a discrete Harmonic analysis bound. $f \in V$, $\{f_1, f_2, \dots\}$ O.N basis on $V$ with inner product $(\ ,\ )$ Then

$$\|f\|^2 = \sum_{i=1}^{\infty} (f, f_i)^2$$

Plancharel; Fourier coefficients

First I'll prove this using Martingale differences then connect it up with Fourier.

**Special cases:**

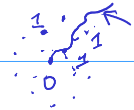
1) If we ~ Exp(1) iid proved by Pemantle Peres.

2) In the percolation case $Var(T(0,x)) = O(\text{constant})$ ← when the minimum edge at percolation

Zhang

3) In $d \geq 3$ its known Bernoulli $\{0,1\}$. (Zhang)

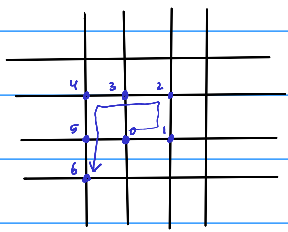
4) I think proofs are easy in LPP.




$I = 1$ mostly having ones with few exceptions

We give the proof where $\tau_e \sim$ Bernoulli $(p)$ $p < p_c$

$\tau_e \in \{0, 1\}$




Enumerate edges in the lattice

$\mathbb{P}(\tau_e = 1) = p$ Need $F(0) = \mathbb{P}(\tau_e = 0) < p_c$

let $T = \tau(0, x)$ where $x$ is some fixed lattice point

form the filtration

$\mathcal{F}_k = \sigma(\tau_{e_1}, \dots, \tau_{e_k})$ (sigma algebra generated by the 1st k edge wts)

$\Delta_k = \mathbb{E}[T \mid \mathcal{F}_k] - \mathbb{E}[T \mid \mathcal{F}_{k-1}]$ (Doob Martingale)

$$\boxed{\mathrm{Var}(T) = \sum_{i=1}^{\infty} \mathbb{E}[\Delta_k^2]} \quad (\#1)$$

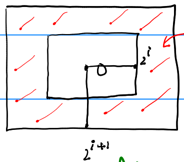
(we've shown this by showing $\underbrace{\mathbb{E}[\Delta_k \Delta_j] = \delta_{kj}}$ ) orthogonality of martingale differences.

Strategy: We will show that

$$\mathbb{E}[\Delta_i^2] \geq \underbrace{p(1-p)}_{\frac{1}{2}} \underbrace{\mathbb{P}(\tau_{e(i)} = 1 \text{ and } e(i) \in \overline{GEO(0,x)})^2}_{\mathbb{P}(F_i)^2}$$

$\mathbb{P}(F_i)^2$ ← Fourier walsh expansion

We will divide up the sum (#1) over various finite subsets I



annuli

$A_{i+1} = R_{i+1} \setminus R_i$

$R_i = [-2^i, 2^i]^2$ (Rectangle of side length $2^i$)

a sum in an annulus $A_i$

$$\sum_{j \in A_i} \mathbb{P}(F_j)^2 = |A_i| \underbrace{\frac{1}{|A_i|} \sum_{j \in A_i} \mathbb{P}(F_j)^2}$$

looks an expectation

$\mathbb{E}[A^2] \geq (\mathbb{E}[A])^2$ Jensen

$$\geq |A_i| \left( \frac{1}{|A_i|} \sum_{j \in A_i} \mathbb{P}(F_j) \right)^2$$

$$= \frac{1}{|A_i|} \left( \sum_{j \in A_i} \mathbb{P}(F_j) \right)^2 = \frac{1}{|A_i|} \mathbb{E}\left[ \#\text{ of edges in } \overline{GEO}(0,x) \text{ when crossing } A_i \text{ that have value } 1 \right]^2$$

I am going to write $T =$ as a sum of martingale differences. I am going to arrive at a bound that is extremely similar to the fourier-walsh based bound.

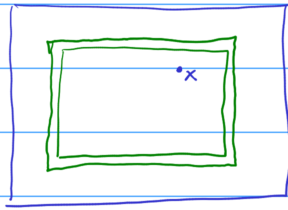
$$\mathrm{Var}(f) \geq \frac{1}{2} \mathbb{P}(e(i) \in \overline{GEO}(0,x), \hat{\omega}_i = 2)$$

$\int T \, dF_k \, dF_{k+1} \cdots dF_n$ $(\tau_{e(1)} \cdots \tau_{e(k)})$

$\int T \, dF_{k+1} \, dF_{k+2} \cdots dF_n$ $(\tau_{e(1)} \cdots \tau_{e(k)})$

$$T = \sum_{k=1}^{n} \Delta_k$$

$$\mathbb{E}[T^2] = \sum_{1 \le k,j}^{n} \mathbb{E}[\Delta_k \Delta_j]$$



$|x|_1 < 2L$

L

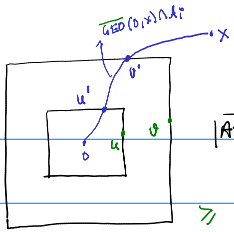
dyadic decomposition

$\{1, \dots, n\}$

↓

$\{a_1, \dots, a_n\}$

↳ $\frac{1}{n} \sum a_i^2 = \mathbb{E}[A]$




Union over all geodesics from 0 to x.

$$\frac{1}{|A_i|}\,\mathbb{E}\Big[\underbrace{W}_{\text{weights}}\big(\overline{GEO}(0,x)\cap A_i\big)\Big]^2$$

$$\geq \frac{1}{|A_i|}\,\mathbb{E}\Big[\min_{\substack{u,v\\ \Gamma_{u\to v}}} W(\Gamma_{u\to v})\Big]^2$$

$$\sum_{j\in A_i}\mathbb{P}(F_j) = \mathbb{E}\Big[\sum_{j\in A_i}\mathbb{1}_{\{F_j\}}\Big] = \mathbb{E}\Big[\sum_{j\in A_i}\mathbb{1}_{\{j\in\overline{GEO}(0,x),\,\omega_j=1\}}\Big]$$

minimum over passages times from u to v

$$\mathrm{Var}(T) \geq \overbrace{p(1-p)}^{C_p}\sum_{i=1}^{\infty}\mathbb{P}(F_i)^2$$

$$\geq C_p\sum_{i=1}^{\log_2\frac{|x|}{3}}\sum_{j\in A_i}\frac{1}{|A_i|}\,\mathbb{E}\Big[\min_{\substack{u,v\\ \Gamma_{u\to v}}} W(\Gamma_{u\to v})\Big]^2 \qquad \text{— (\#3)}$$

Broke it up into a sum over annuli

$u\in\partial R_{i-1}$, $v\in\partial R_i$

Recall Kesten's lemma: if $\mathbb{P}(\tau_e=0)<p_c$ then $\exists\, a, C$

$$\mathbb{P}(\exists\ \Gamma \text{ of length } n \text{ starting at } 0 \text{ st } W(\Gamma)<an) \leq e^{-Cn}$$

→ weight of $\Gamma$ being too small is exponentially unlikely.

For fixed $u,v$

$n$ "length of the path from u to v"

$$\mathbb{P}\big(W(\Gamma_{u\to v}) < a\,2^i\big) \leq e^{-C2^i} \qquad \text{← trivial manipulations.}$$

There are $(2^i)^2$ possibilities for $u$ and $(2^{i+1})^2$ possibilities for $v$

Thus $\mathbb{P}\Big(\bigcup_{u,v} W(\Gamma_{u\to v}) < a2^i\Big) \leq C_1 2^{4i} e^{-C2^i} \leq C_2 e^{-C_3 2^i}$ — #3a

$$\sum_{u,v}\mathbb{P}\big(W(\Gamma_{u\to v})<a2^i\big) \leq \sum_{u,v} e^{-C^{2i}}$$

polynomial

exponential

"The passage time across $A_i$ cannot be too <u>SMALL</u>"

POLL: (Complete the sentence)

A) SMALL

B) LARGE

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > u)\,du$$

Thus,

$$\mathbb{E}\Big[\min_{\substack{u,v\\ \Gamma_{u\to v}}} W(\Gamma_{u\to v})\Big] = \int_0^\infty \mathbb{P}\Big(\min_{\substack{u,v\\ \Gamma_{u\to v}}} W(\Gamma_{u\to v}) \geq t\Big)\,dt$$

$$\mathbb{P}\Big(\bigcap_{u,v} \min_{\Gamma_{u,v}} W(\Gamma_{u,v}) \geq u\Big)$$

taking complement

$$= \int_0^\infty \Big(1 - \mathbb{P}\Big(\bigcup_{u,v} W(\Gamma_{u\to v}) < t\Big)\Big)\,dt$$

$t = a2^i$

$$\int_0^{a2^i} + \cancel{\int_{a2^i}^\infty}$$

drop to get a lower bound.

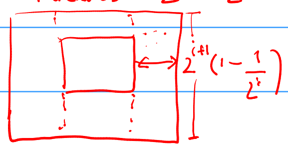
$$\geq \int_0^{a2^i} \Big(1 - \mathbb{P}\Big(\bigcup_{u,v} W(\Gamma_{u\to v}) < a2^i\Big)\Big)\,dt$$

$$\geq \underbrace{a2^i}_{\text{domain of integration}}\big(1 - C_2 e^{-C_3 2^i}\big) \geq \underbrace{C_4 2^i}_{\text{NOT TOO SMALL.}} \quad (\text{using } \#3a)$$

Let's get back to (#2):

$$\mathrm{Var}(T) \geq C_p \sum_{i=1}^{\overbrace{\log_2 \frac{|X|_1}{2}}^{\#\text{ of annuli } \log_2 \frac{|X|_1}{2}}} \sum_{j\in A_i} \frac{1}{|A_i|}\,\mathbb{E}\Big[\min_{\substack{u,v\\ \Gamma_{u\to v}}} W(\Gamma_{u\to v})\Big]^2$$

$A_i$ is an annulus at radius $2^i$ and it has thickness $2^{i+1} - 2^i$




$$\geq C_p \sum_{i=1}^{\log_2 \frac{|X|_1}{2}} \frac{1}{2^{2i}}\,(C_4 2^i)^2$$

$2^{2i}$ ← # of points in the annulus $A_i$

$$\underbrace{\geq C_5 \log_2 |X|_1}$$

OK, now left to prove the following.

$$\mathbb{E}[\Delta_i^2] \geq p(1-p)\,\mathbb{P}(F_i)^2$$

dyadic decomp Kesten's bound

where $\Delta_i = \mathbb{E}[T \mid \Sigma_i] - \mathbb{E}[T \mid \Sigma_{i-1}]$

(Martingale increment of course)

$$\mathrm{Var}(T) = \sum \mathbb{E}[\Delta_i^2]$$

$$F_i = \{\tau_{e(i)} = 1 \text{ and } i \in \overline{GEO}(0,x)\}$$

$$\Sigma_i = \sigma(\tau_{e(1)}, \dots, \tau_{e(i)})$$

$$\Sigma_{i-1} = \sigma(\tau_{e(1)}, \dots, \tau_{e(i-1)})$$

"Analogous to measuring the effect of changing the $i^{th}$ variable"

$$\mathrm{Var}(T) = \sum_{\substack{S\subseteq[n]\\ S\neq\emptyset}} \hat{f}_S^{\,2} \quad (\text{Fourier Walsh})$$

Let $T_i^\delta$ for $\delta = 0,1$ be the passage time $T$ with edge $\tau_{e(i)} = \delta$ and all other weights being the same.

$\tau_{e(i)} \in \{0,1\}$

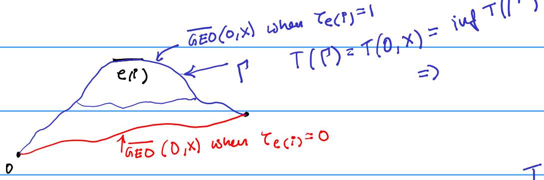
Let $H_i = T_i^1 - T_i^0$

$T(e(1), e(2) \dots, 1, e(i+1) \dots) - T(e(1), e(2) \dots, 0, e(i+1) \dots)$

$H_i$ does not depend on the rv $e(i)$

$$H_i = \begin{cases} 1 & \text{if } i \in \overline{GEO}(0,x) \text{ when } \tau_{e(i)} = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{— (\#4)}$$

Why is that?




$T(\Gamma) = T(0,x) = \inf T(\Gamma')$

$\Rightarrow$

Can it happen that $e(i) \in \overline{GEO}(0,x)$ when $\tau_{e(i)} = 1$ $\Rightarrow$ $T_i^1 - T_i^0 = 1$

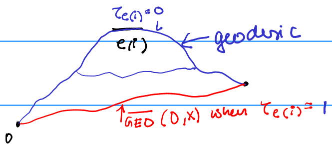
But $e(i) \notin \overline{GEO}(0,x)$ when $\tau_{e(i)} = 0$

That's not possible since $T_i^1 = T_i^0$ on the red path.

| | | |
|---|---|---|
| $e(i) \in \overline{GEO}(0,x)$ | $\tau_{e(i)} = 1$ | $H_i = 1$ |
| $e(i) \in \overline{GEO}(0,x)$ | $\tau_{e(i)} = 0$ | $H_i = 0$ |
| $e(i) \notin \overline{GEO}(0,x)$ | $\Rightarrow$ | $H_i = 0$ |

What if $e(i) \in \overline{GEO}(0,x)$ when $\tau_{e(i)} = 0$

$e(i) \notin \overline{GEO}(0,x)$ when $\tau_{e(i)} = 1$




In this case one can check that $T_i^1 = T_i^0$ on the red path $\Rightarrow$ $H_i = 0$

So again (#4) is verified.

Now we repeat the argument from 50 years.

$$\Delta_i = E[T \mid \Sigma_i] - E[T \mid \Sigma_{i-1}]$$

$$= E[T_i^0 + H_i \tau_{e(i)} \mid \Sigma_i]$$

$$- E[T_i^0 + H_i \tau_{e(i)} \mid \Sigma_{i-1}]$$

$$T = T_i^0 + H_i \tau_{e(i)} = \begin{cases} T_i^0 + 1 & \text{if } i \in \overline{GEO}(0,x)\ \tau_{e(i)} = 1 \\ T_i^0 & \text{otherwise} \end{cases}$$

is independent of the value of $\tau_{e(i)}$

Easy to check.

$\tau_{e(i)} \in \Sigma_i = \sigma(\tau_{e(1)}, \dots \tau_{e(i)})$

does not depend on $\tau_{e(i)}$

$$= \mathbb{E}[T_i^0 \cancel{\mid \Sigma_i}] + \tau_{e(i)}\, \mathbb{E}[H_i \mid \Sigma_i]$$

$$- \mathbb{E}[T_i^0 \cancel{\mid \Sigma_{i-1}}] - \mathbb{E}[\tau_{e(i)}]\, \mathbb{E}[H_i \mid \Sigma_{i-1}]$$

$$\mathbb{E}[T_i^0 \mid \Sigma_{i-1}] = \mathbb{E}[T_i^0 \mid \Sigma_i]$$

$$= (\tau_{e(i)} - E[\tau_{e(i)}])\, \mathbb{E}[H_i \mid \Sigma_{i-1}]$$

again using indep. $H_i$ and $\tau_{e(i)}$

$T_i^0$ also does not depend on $\tau_{e(i)}$

$$\mathbb{E}[H_i \mid \Sigma_i] = \mathbb{E}[H_i \mid \Sigma_{i-1}]$$

$$\mathbb{E}[\Delta_i^2] = \mathbb{E}\left[(\tau_{e(i)} - \mathbb{E}[\tau_{e(i)}])^2\right] \mathbb{E}\left[\mathbb{E}[H_i \mid \Sigma_{i-1}]^2\right]$$

$$\Rightarrow E[\Delta_i^2] = \mathrm{Var}(\tau_{e(i)})\, \mathbb{E}\left[\mathbb{E}[H_i \mid \Sigma_{i-1}]^2\right]$$

$$\geq p(1-p)\, \mathbb{E}[H_i]^2$$

Jensen's inequality and Tower property.

$$\mathbb{E}[H_i] = 1\, \mathbb{P}(\tau_{e(i)} = 1,\ e(i) \in \overline{GEO}(0,x))$$

$$= \mathbb{P}(F_i)$$

- This is one place where we lose something (Jensen)
- Dyadic decomposition and Jensen.

$$\mathbb{E}\left[\mathbb{E}[H_i \mid \Sigma_{i-1}]^2\right] \geq \mathbb{E}\left[\mathbb{E}[H_i \mid \Sigma_{i-1}]\right]^2$$

freeze, unfreeze

and we're done.

Remarks:

$\mathrm{Var}(T(0,x)) \geq C \log|x|$ $\qquad d = 2$

Truth $\qquad \approx |x|^{2\chi} \qquad \chi = \frac{1}{3}$ (fluctuations exponent)

$|x|^{\epsilon}$ (That would be a BIG contribution)

Worst bound we obtained above

Fourier-Walsh way: $\quad \mathrm{Var}(T(0,x)) = \sum \hat{f}_S^2 \geq \sum \hat{f}_{\{i\}}^2$ $\qquad f = T(0,x)$, $\hat{f}$ = fourier coefficients

$\hat{f}_{\{i,j\}}$

Question (research): Can you compute these Fourier coefficients? Or reinterpret them in some nice way and relate them to some property of the geodesic?