Lec 14 Kesten's general BK inequality, and its application to FPP geodesic length.

(Kesten, Aspects Theorem 4.8) Let

$\{A(k,i) : k,i \geq 1\}$ be a family of inc. events on $(\Omega, \mathbb{P})$

$\{A'(k,i) : k,i \geq 1\}$ be another family on $(\Omega, \mathbb{P}')$

For each fixed i Let

1) $\{A(k,i) : k \geq 1\}$ have the same distribution (meas $\mathbb{P}$)

as $\{A'(k,i) : k \geq 1\}$ (meas $\mathbb{P}'$)

2) $\{A'(k,i) : k \geq 1\}$ are independent families under $\mathbb{P}'$

Then fix some $n(k) < \infty$. for each $k = 1, 2, \ldots$

each row can have a different # of columns

$$\begin{matrix} A'(1,1), & A'(1,2) & \cdots & A'(1, n(1)) \\ A'(2,1), & A'(2,2) & \cdots & A'(2, n(2)) \\ \vdots & & & \end{matrix}$$

] independent rows

need not be independent within RED

Blue is indep of green.

Then

$$\mathbb{P}\left(\bigcup_{k \geq 1} A_{k,1} \circ A_{k,2} \circ \cdots A_{k,n(k)} \right)$$

union is over the column

$\mathrm{Var}(T(0,x)) \geq C \log|x|$,

$\mathrm{Var}(T(0,x)) \leq \dfrac{C|x|}{\log|x|}$

$d=2$

$\mathrm{Var}(T(0,x)) \approx C|x|^{2/3}$

"Exponential estimate on percolating paths"

↳ BK inequality.

↳ Discrete Fourier-analysis

Arian is going tell us about some Harmonic analysis inequalities related to this.

POLL: Have you seen basic Harmonic analysis on $[0,1]$, l.e. $\{\sin nx, \cos nx\}_{n=0}^{\infty}$?

YES OR NO

$$\begin{matrix} A(1,1) & A(1,2) & \cdots \\ A(2,1) & A(2,2) & \cdots \end{matrix}$$

$k=1$ in original BK

$$\mathbb{P}(A_{1,1} \circ A_{1,2} \cdots \circ A_{1,n}) \leq \mathbb{P}\left(\bigcap_{i=1}^{n} A'(1,i)\right)$$

BK generalized

intersection over independent columns.

$$\leq \mathbb{P}\Big(\bigcup_{k\geq 1} \overbrace{\bigcap_{i=1}^{n(k)} A'(k,i)} \Big) \qquad -(\#12)$$

Then using the union bound we usually write this as

$$\leq \sum_{k\geq 1} \mathbb{P}\Big(\bigcap_{i=1}^{n(k)} A'(k,i) \Big)$$

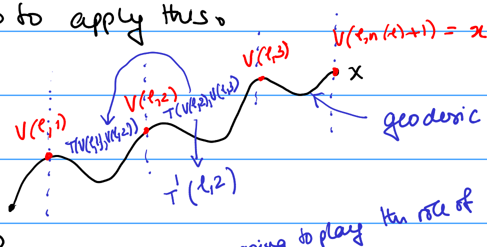
independence

$$= \sum_{k\geq 1} \prod_{i=1}^{n(k)} \mathbb{P}\big(A'(k,i) \big) \qquad \text{using independence.}$$

$-$ #1

large but finite box contains all these paths Γ

How to apply this.



geodesic from 0 to x

ℓ is going to play the role of k we used previously.

For each ℓ, $\{V(\ell,i)\}_{\ell,i}$ are sets of vertices.

$\Pi_\ell(0,x,t) = \{ \Gamma : \Gamma$ is a path from 0 to x that goes through $V(\ell,1),\ldots V(\ell,n(\ell))$ and $T(\Gamma) < t \}$

$\Pi_\ell(0,x,t)$ is an event (on which the passage time is small on a collection of paths)

$$\mathbb{P}\Big(\bigcup_{\ell\geq 1}\Pi_\ell(0,x,t)\neq\emptyset\Big)$$

$= \mathbb{P}\big(\exists$ a path Γ that passes through one of the collections over of vertices $V(\ell,1),\dots V(\ell,n(\ell))$ such that $T(\Gamma)<t\big)$

$$\leq \overbrace{\sum_{\ell\geq 1}}^{\text{union bound}} \mathbb{P}\Big(\sum_{j=0}^{n(\ell)} \overbrace{T'(\ell,j)}^{\text{independent rvs over } j \text{ / columns}} < t\Big)$$

We know how to analyze independent sums.

where $\{T'(\ell,j)\}_{j=1}^{n(\ell)}$ are independent and

$$T'(\ell,j) \overset{d}{=} T\big(V(\ell,j),V(\ell,j+1)\big)$$

Remark: We know how to analyze independent SUMS!

Proof :

$\mathbb{P}$($\exists$ a self avoiding path, Γ that passes through one of the collections of vertices $V(\ell,1),\dots V(\ell,n(\ell))$ such that $T(\Gamma) < t$)

POLL: Are you all more or less with me ?

YES OR NO

we can write this event as $\left\{\bigcup_{\ell \geq 1} \Pi_\ell(0,x,t) \neq \emptyset\right\} = \left\{\begin{array}{l}\exists \text{ a } \Gamma, \text{ passing through } \\ V(\ell,1)\dots V(\ell,n(\ell)) \text{ st } \\ T(\Gamma) < t\end{array}\right\}$

$$\underbrace{\bigcup_{\ell \geq 1} \bigcup_{m \geq 1}} A(m,\ell,1) \circ \dots \circ A(m,\ell,n(d)))$$

$T(\Gamma) < t$

$r \geq \sum_{i=1}^{n(\ell)} t(m,\ell,i)$

There are two indices instead of 1, but this is still in the BK setting.

where

$$A(m,\ell,i) = \{T(V(\ell,i), V(\ell,i+1)) < \underbrace{t(m,\ell,i)}_{\text{rational \#.}}\}$$

index all possible nonnegative rational numbers.

$$\sum_{i=1}^{n(\ell)} t(m,\ell,i) < t$$

So basically you're picking a dense collection of rational #s, so that you can look for all the ways in which

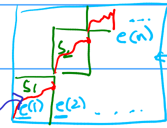
$$\sum_{i=1}^{n(\ell)} T(V(\ell,i), V(\ell,i+1)) < t$$

Fix m, ℓ

$A(m,\ell,1) \circ \cdots \circ A(m,\ell,n(\ell))$ n-fold disjoint occurrence.

$= \{\omega \mid S_i \subset \{e(1), \cdots\},\ S_i \cap S_j = \phi$

$C(\omega, S_i) \subset A(m,\ell,i)\}$



Enumeration of the edges.

$C(\omega^{\{0,1\}^n}, S) = \{\omega' \mid \omega'_i = \omega_i \text{ on } S\}$

$= \{\exists$ a self avoiding path Γ, such that the segment from $V(\ell,i)$ to $V(\ell,i+1)$ takes time less that $< t(m,\ell,i)$ $i=1,\ldots,n(\ell)\}$

the self-avoiding path γ specifies the subset of edges S_k

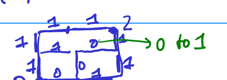
What are these edges S_k for any configuration ω?

Then BK gives

$\mathbb{P}\left(\bigcup_{m,\ell} A(m,\ell,1) \circ \cdots \circ A(m,\ell,n(\ell))\right)$

represents the collection of vertices $V(\ell,1), \ldots, V(\ell,n(\ell))$

$\{T(V(\ell,i), V(\ell,i+1)) < t(m,\ell,i)\}$ ↓1



$\leq \sum_\ell \mathbb{P}\left(\bigcup_m A(m,\ell,1) \circ \cdots \circ A(m,\ell,n(\ell))\right)$

BK inequality

$\leq \sum_\ell \mathbb{P}\left(\bigcup_m \bigcap_{i=1}^{n(\ell)} \overbrace{A'(m,\ell,i)}^{\text{independent over } i}\right)$ (over distinct sections of the path)

rationals were designed so $t(m,\ell,i)$ summed smaller than t

$= \sum_\ell \mathbb{P}\left(\sum_{i=1}^{n(\ell)} T'(\ell,i) < t\right)$

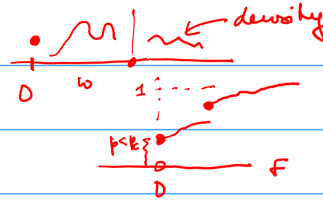
<u>POLL</u>: Are $A(m,\ell,i)$

A increasing events

B decreasing events.

$(1_{A(m,\ell,i)}$ an increasing fn of $\omega)$

Back to Kesten's Lemma (Using the BK inequality)



If $F(0) < p_c$ then $\exists\, C, a > 0$ st

$\mathbb{P}\big(\exists$ a self-avoiding Γ starting at 0 st $|\Gamma| > n$,

$T(\Gamma) < an\big) \leq e^{-Cn}$

small constant.

(C large, corresponds to e' length)
a is small corresponds to passage time.

Q: Do you know how to prove

$$\mathbb{P}\left(\lim \frac{T(0,nx)}{n} < a\right) = 0$$

YES OR NO

Using this theorem,
Or $\mu(x) \geq a$ so for any $x \neq \vec{0}$

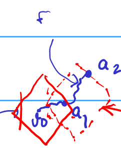
$$\lim_{n\to\infty} \mathbb{P}\left(\begin{array}{l}\exists\, \Gamma \cdots |\Gamma| > n \\ \frac{T(\Gamma)}{n} < a\end{array}\right) \leq e^{-Cn} \to 0$$

$\sum_n \mathbb{P}(A_n) < \infty$

$\mathbb{P}(A_n \text{ i.o}) = 0$

Pf: Fix any Γ of length n. Then label its points $\Gamma = (v_0, \ldots v_n)$. Define

Fix some M.



box of size M

Each of them red boxes has $(2M+1)^d$

$[-M,M]^d$

let $\overbrace{t(0)}^{\text{times}} = 0 \qquad \overbrace{a_0}^{\text{vertex}} = v_0$

let $t(1) = \inf\{t > 0 \mid |v_t - a_0| = M\}$

$a_1 = v_{t(1)}$

(Skeleton of points along Γ. Help us control the "entropy". Peierls, 2D Ising model)

(Inductively define $t(k)$ = time at which you leave the box around a_{k-1},

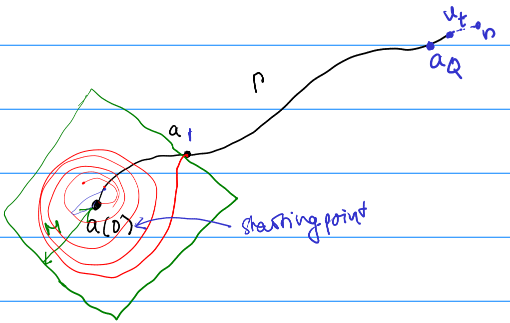
a_k = the vertex at which you leave the box.)

$$t(k) = \inf\{t > 0 : |v_t - a_{k-1}|_1 = M\}$$

$$a_k = v_{t(k)}$$

Let Q be st $|a_Q - v_t|_1 < M \quad \forall\, n > t > Q$

(final time)



In each L^1 ball of size M around $a(k)$

there are $(2M+1)^d$ Lattice points.

Since self avoiding. $Q \geq \dfrac{n}{(2M+1)^d} = O(n)$ many skeleton points.

constant

$\vec{a} = (a_1, a_2, \ldots, a_Q)$

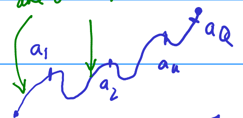
Now sum over all points in the skeleton.

$V(e,1) \ldots V(e,Q)$ → collection of fixed vertices

$$\mathbb{P}\Big(\underbrace{\bigcup_{Q \geq \frac{n}{(2M+1)^d}}^{n}}_{\mathbb{P}(\cup A_i) \leq \sum_i \mathbb{P}(A_i)} \bigcup_{\vec{a}} \Gamma \text{ passes through } a_1, \ldots, a_Q \text{ and } T(\Gamma) < an \Big) = \mathbb{P}\big(\exists\, \Gamma,\ |\Gamma| = n, \text{ st } T(\Gamma) < an\big)$$

where $\vec{a} = (a_1, \ldots, a_Q)$ such that $|a_{i+1} - a_i|_1 = M$

are on disjoint sets of edges



$T(\Gamma) = T(\Gamma_{0,a_1}) + T(\Gamma_{a_1,a_2}) \ldots$

Using the BK inequality and the union bound

$$\leq \sum_{Q \geq \frac{n}{(2M+1)^d}} \sum_{\vec{a}} \mathbb{P}\Big(\sum_{i=0}^{Q-1} T'(a_i, a_{i+1}) < Cn\Big)$$

this is the sum over e in the BK inequality.

The times over the disjoint sections of the path have been replaced by independent variables $\{T'(a_i, a_{i+1})\}_{i=1}^{Q}$

$$\sum_{\vec{a}} \mathbb{P}\Big(\sum_{i=0}^{Q-1} T'(a_i, a_{i+1}) < Cn\Big)$$

$$\Big\{\sum_{i=0}^{Q-1} T'(a_i, a_{i+1}) < Cn\Big\} = \Big\{ e^{\lambda \sum_{i=0}^{Q-1} T'(a_i, a_{i+1})} < e^{\lambda Cn} \Big\}$$

$$= \sum_{\vec{a}} \mathbb{P}\Big(e^{\lambda \sum_{i=0}^{Q} T'(a_i, a_{i+1})} < e^{\lambda Cn} \Big)$$

Chernoff trick. $\lambda > 0$

$$\leq \sum_{\vec{a}} \mathbb{E}\Big[e^{-\lambda \sum_{i=0}^{Q} T'(a_i, a_{i+1})} \Big] e^{\lambda Cn}$$

$$\mathbb{P}(X < a) = \mathbb{E}\big[\mathbb{1}_{\{X < a\}}\big] \qquad X > 0$$

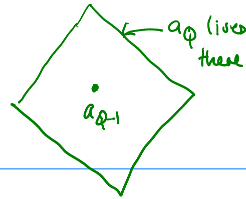
$$\leq \mathbb{E}\Big[\frac{a}{X} \mathbb{1}_{\{X < a\}}\Big]$$ remove restriction

$$\leq \mathbb{E}\Big[\frac{a}{X}\Big]$$

$$= \sum_{\vec{a}} \mathbb{E}\Big[\prod_{i=0}^{Q-1} e^{-\lambda T'(a_i, a_{i+1})} \Big] e^{\lambda Cn} \quad \text{—(\#1)}$$

independence ↪ $\prod_{i=0}^{Q-1} \mathbb{E}\big[e^{-\lambda T'(a_i, a_{i+1})}\big]$

(Peeling)



Now you "peel the layers". It's not explained so well in the text book.

$$\sum_{a_1}\sum_{a_2}\sum_{a_3}\cdots\sum_{a_Q} e^{\lambda C n}\prod_{i=0}^{Q-2}\mathbb{E}\left[e^{-T'(a_i,a_{i+1})}\right]\mathbb{E}\left[e^{-T'(a_{Q-1},a_Q)}\right]$$

$$\mathbb{E}\left[e^{-T'(0,a_Q-a_{Q-1})}\right]$$

Fix $a_1 \cdots a_{Q-1}$ and sum over a_Q

Thus

$$\sum_{a_1}\sum_{a_2}\cdots\sum_{a_{Q-1}} e^{\lambda C n}\prod_{i=0}^{Q-2}\mathbb{E}\left[e^{-\lambda T'(a_i,a_{i+1})}\right]\underbrace{\sum_{|a|_1=M}\mathbb{E}\left[e^{-\lambda T'(0,a)}\right]}_{\text{does not depend on } a_1\cdots a_{Q-1},\,a_Q}$$

Sum over a_Q.

Finished peeling one layer of the onion.

$$= e^{\lambda C n}\left(\sum_{|a|_1=M}\mathbb{E}\left[e^{-\lambda T'(0,a)}\right]\right)^{Q}$$

"This is the power of independence"

for large λ

$$e^{-\lambda T'(0,a)} \approx \mathbb{1}_{\{T'(0,a)=0\}} + \mathbb{1}_{\{T'(0,a)>0\}}\, e^{-\lambda T'(0,a)} \to 0 \text{ as } \lambda\to\infty$$

connection to 0,1 percolation comes in.

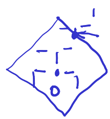
OK, let's think about

$$\sum_{|a|_1=M}\mathbb{E}\left[\mathbb{1}_{\{T'(0,a)=0\}}\right] = \sum_{|a|_1=M}\mathbb{P}(T(0,a)=0)$$

(#2)

"all points $a\in\{|x|_1=M\}$ that are connected to the origin by a path of 0s"

$$\leq \mathbb{E}\left[\underbrace{\text{size of open, connected cluster}}_{\text{that passes through the origin}}\right]$$



$\mathbb{P}$(an edge being open) $= F(0) < p_c$

critical prob.

$W := \{ v \in \mathbb{Z}^2 : 0 \to v$ by a path of 0s $\}$

= "percolation cluster passing through the origin"

$\mathbb{E}[W] < \infty$ if $F(0) < p_c \to \mathbb{P}($ edge is open or $0) < p_c$

↳ Duminil-Copin and Tassion (2015)

↳ Grimmett, Theorem 5.75. (Textbook on percolation)

Return to (#2) and notice that it is

$\mathbb{E}[W \cap \{x : |x|_1 = M\}]$

$$\mathbb{E}[W] = \sum_{k=1}^{\infty} \mathbb{E}[W \cap \{x : |x|_1 = k\}] < \infty$$

So you can choose M st

M^{th} term in the sum is small.

$$\mathbb{E}[W \cap \{x : |x|_1 = M\}] \leq \frac{1}{2}$$

$$\sum_{|a|_1 = M} \mathbb{E}\left[e^{-\lambda T'(0,a)}\right] = \overbrace{\sum_{|a|_1=M} \mathbb{E}\left[\mathbb{1}_{\{T'(0,a)=0\}}\right]}^{\leq 1/2 \ (\#3a)} + \underbrace{\sum_{|a|_1=M} \mathbb{E}\left[e^{-\lambda T'(0,a)} \mathbb{1}_{\{T'(0,a)>0\}}\right]}_{(\#3b)}$$

I can also choose this to be small for large enough λ.

Thus returning to (#1)

$$= e^{\lambda C n} \left(\sum_{|a|_1 = M} \mathbb{E}\left[e^{-\lambda T(0,a)}\right] \right)^{Q} \quad (\#3)$$

Once M is chosen, choose λ so large so that

$$e^{-\lambda T'(0,a)} \mathbb{1}_{\{T'(0,a)>0\}} \to 0 \quad \text{as } \lambda \to \infty$$

#3a and #3b

combining

$$\sum_{|a|=M} \mathbb{E}\left[e^{-\lambda T(0,a)}\right] \leq \frac{3}{4}.$$ (Why can you do this?)

we first fixed M

λ has been fixed large

$Q \geq \frac{n}{(2M+1)^d}$ ← proved this previously

Then, $(\#3) \leq e^{\lambda C n}\left(\frac{3}{4}\right)^{Q}$ $\qquad e^{\lambda C n}\left(\sum_{|a|=M} \mathbb{E}\left[e^{-\lambda T(0,a)}\right]\right)^{Q}$ — (#3)

$(e^{\lambda C}) > 1$ by choosing C small

Thus $\leq (e^{\lambda C})^{Q(2M+1)^d}\left(\frac{3}{4}\right)^{Q}$ — (#4)

In fact choose C so small st $e^{\lambda C(2M+1)^d}$ is very small. Then

$$\left(e^{\lambda C(2M+1)^d}\,\frac{3}{4}\right) \leq \frac{7}{8}$$

not much larger than 1

$Q \geq \frac{n}{(2M+1)^d}$

$(\#4) \leq \left(\frac{7}{8}\right)^{Q} \leq e^{-C' n}$ $\qquad e^{-C'} = \left(\frac{7}{8}\right)^{\frac{1}{(2M+1)^d}}$

This is the exponential bound we wanted to prove.

Original probability we started with.

POLL: Were you all more or less with me?

A YES

B NO