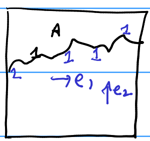
Lec 13 separated occurrence inequalities

Let E^d be the edges on the lattice $\mathbb{Z}^d$ and suppose $e_1, \dots, e_n \in E^d$. Put iid $\{0,1\}$ Bernoulli (p) rvs on all the edges. Let

$\Omega = \{0,1\}^{E^d}$

Let A, B be increasing events and (suppose they depend on) depend on $\omega = (\omega(e_1), \dots, \omega(e_n))$

Ex: (from percolation) Let B be an $N \times N \cdots \times N$ box. ($N \times N$) We say $\exists$ an e_1 open crossing if $\exists$ a path of 1 weights from the "left side" of the box to the "right side" (x_1 coordinate 0 to x_1 coordinate N). Let A be this event.



A is an increasing event.

$2\chi - 1 = \xi$ is

$\leftarrow \mathrm{Var}(T(0,nx)) \sim n^{2\chi}$

This had been conjectured by the physicists. (1986?)

↳ S. Chatterjee (2013 ish?) ← Stanford.

↳ Annals of Math.

Auffinger & Damron (who improved Chatterjee's proof) → Georgia Tech

→ Ann. of Prob.

$\mathrm{Var}(T(0,nx)) \le \frac{Cn}{\log n}$ as $n \to \infty$ for fixed x.

$n^{2\chi} = n^{2/3}$ in $d=2$

$n^{1-\epsilon}$ is a major open problem.

$\mathrm{Var}(T(0,nx)) \ge C$ at most

$\ge C \log n$ (Newman-Piza)

Is going to take me to Kesten's inequality and the BK inequality.

$\mathbb{P}(A \cap B) \ge \mathbb{P}(A)\mathbb{P}(B)$

$\mathbb{P}(A \circ B) \le \mathbb{P}(A)\mathbb{P}(B)$ BK.

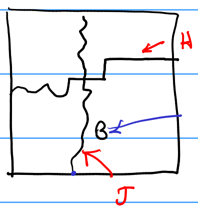
↑ new symbol (separated occurrence together)

Suppose B is the event that there is an e_2 crossing.

How do I represent the event that A and B BOTH occur (like an intersection $A \cap B$), but do so with any edge overlap?

↳ The paths that I get should not USE the same edges.

A



Such events are important in percolation.

This is called a disjoint occurrence. The BK inequality shows that

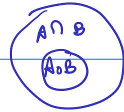
$$\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\,\mathbb{P}(B).$$

Now suppose we are given that B occurs. Then B has "used up" some edges and there are fewer edges for A to use.

So we must have

A disjoint B

$\mathbb{P}(A \circ B \mid B) \leq \mathbb{P}(A)$ definition of conditional probability



$$\frac{\mathbb{P}((A \circ B) \cap B)}{\mathbb{P}(B)} \leq \mathbb{P}(A)$$

$$= \mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B)$$

(The intuition for WHY something like this is true)

let us define disjoint occurrence

Each $\omega = (\omega(e_1), \ldots, \omega(e_n)) \in \{0,1\}^n$ (0 1 0 ⋯ 0 1)

can be defined using $K(\omega) \subset \{e_1, \ldots, e_n\}$

the subset of edges on which $\omega(e_i) = 1$

$$K(\omega) = \{ e_i \mid \omega(e_i) = 1, \ i = 1, \ldots, n \}$$

POLL

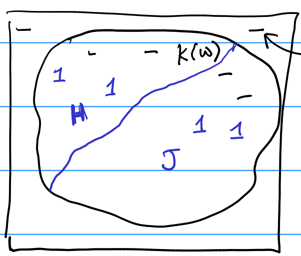
Are you with me?

YES OR NO

all possible 0,1 configs in the set of edges

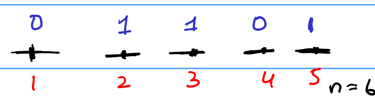
$$A \circ B = \{ \omega \in \{0,1\}^n \mid \exists H \subset K(\omega) \text{ st } K(\omega') = H \Rightarrow \omega' \in A, \text{ and } \exists J \subset K(\omega) \setminus H \text{ st } K(\omega'') = J \Rightarrow \omega'' \in B \}$$

I will modify this in a second



edges $= \{e_1, \ldots, e_n\}$

$K(\omega) = \{2, 3, 5\}$



$H = \{2,3\}$ $J = \{5\}$

Other ω' $K(\omega') = H$

and $\exists J \subset K(\omega) \setminus H$

ω' being 1 on H is enough to force A, ω'' being 1 on J is enough to force B.

Ex: Come up with a definition for $A \circ B \circ C$ and show that $A \circ B \circ C = A \circ (B \circ C)$ (associative property)

Grimmett in this book called Percolation claims it's true

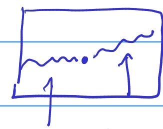
But Auffinger Damron Hanson says it's not always true (associativity)

Theorem (B.K) Let A, B be increasing events on $\{0,1\}^n$.

Then $\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\,\mathbb{P}(B)$

$A = \{\exists\ e_1 \text{ crossing}\}$

$B = \{\ "\ e_2\ "\ \}$



disjoint sections of geodesics are going to behave independently.

It follows that if $A_1, \dots, A_k$ are increasing events then

$$\mathbb{P}(A_1 \circ \cdots \circ A_k) \leq \prod_{i=1}^{k} \mathbb{P}(A_i)$$

There are some subtleties:

$A \circ B$ can be defined as follows: (for increasing events)

Recall $\Omega = \{0,1\}^n$

$K_1(\omega) = \{i : \omega(i) = 1\}$

$\hat{C}(\omega, H) = \{\omega' : K(\omega') = H\}$ This is just one ω' (all the other coordinates are 0) singleton.

I want it to be a cylinder event (but it's not, as defined).

implies A

$$A \circ B = \{ \omega : \exists H \subset K(\omega) \text{ st } \hat{C}(\omega, H) \subseteq A, \ \hat{C}(\omega, \underbrace{K(\omega) \setminus H}_{J}) \subseteq B \}$$

implies

BK conjectured a more general theorem.

$\overbrace{0 \quad 1 \quad 1}^{H}$			0	1	$= \omega$
1	2	3	4	5	

let $\overbrace{C(\omega, H)}^{\text{cylinder set}} = \{ \omega' : \omega'_i = \omega_i \quad i \in H \}$

$C(\omega, H) = \{ \omega' \mid \omega'_1 = 0, \ \omega'_2 = 1, \ \omega'_3 = 1 \}$

proper cylinder.

let

larger than $\hat{C}$ that we had define

$$A \square B = \{ \omega : \exists H \text{ st } C(\omega, H) \subseteq A \text{ and } C(\omega, H^c) \subseteq B \}$$

The difference here is that we match on any subset of coordinates, not just the coordinates that are 1.

Exercise: when A and B are inc. events. $A \square B = A \circ B$.

Conjecture: (vBK) For any events A and B

$$\mathbb{P}(A \square B) \le \mathbb{P}(A)\mathbb{P}(B)$$

Pete Winkler at Dartmouth.
↳ Erdős #2

Reimer was working on this problem.

Bernoulli $(\frac{1}{2})$.

~2000s he solves this conjecture!

1988 something like that

(and not just increasing ones).

History : The BK conjecture was made in the 90s.

- It has a combinatorial "feel"
- Many people tried.
- Reimer had a small job in a little college near Rutgers. He had been working on this problem for a few years and showed results to a famous combinatorialist at Rutgers. (maybe Beck)
- was told it would not work.

(Did P. Winkler told me the story)

- Finally solved it in 2000.

Now called the BKR inequality.

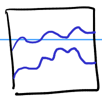
Haven't seen any applications of this though. they haven't seen any good applications of it so far.

POLL : Do you want to see the proof of the BK inequality ?

YES OR NO

Lets prove the theorem while the notation is fresh in our head.

Idea: "doubling edges". Let us build intuition by specializing to events in a box B



e_1 crossing

$A \circ B = \exists$ 2 disjoint open e_1 crossings.

Open path: nearest neighbor path that encounters only 1 edge weights.

A: $\exists$ an open path from left side of Box to the right side

$B = A$

$A \circ B = \exists$ two disjoint open paths from the left to the right.

Let e be an edge in Box and consider some $w \in A \circ B$. You split e $w(e)$ into two edges e' and e'' and put iid Bernoullis on e' and e'' $w(e')$ and $w(e'')$. You look for open paths that use e', and a distinct open path that use e''.

This procedure can only increase $A \circ B$ since the disjoint occurrence only becomes more likely. (it removes the possibility that the two open paths will use the same edge e) and it makes A use e' edges and B use e'' edges

Then replace every edge e in BOX by independent pairs e' and e''. Then eventually $A \circ B$ will become $A' \circ B''$. where A only uses the edges e' and B uses the e'' edges. Thus

Heuristic strategy.

$\mathbb{P}(A' \circ B'') = \mathbb{P}(A')\,\mathbb{P}(B'')$ by independence.

Somehow introduce & use independence.

Proof: Let $(\Gamma, \mathbb{P})$ be a prob. space. Γ "$\{0,1\}^n$", $\mathbb{P}$ Bernoulli$(\frac{1}{2})^{\otimes n}$ product measure

By splitting, we will produce a space

$(\Gamma_1 \times \Gamma_2, \mathbb{P}_1 \times \mathbb{P}_2) = (\{0,1\}^n \times \{0,1\}^n, \mathbb{P} \times \mathbb{P})$

where $\Gamma_i = \Gamma$ and $\mathbb{P}_i = \mathbb{P}$

Call $\mathbb{P}_1 \times \mathbb{P}_2 = \mathbb{P}_{12}$. Let A, B be inc. events. Take

$x \times y \in \Gamma_1 \times \Gamma_2$ I don't care about y

$\mathbb{P}_{12}(A' \circ B_0') = \mathbb{P}(A \circ B)$ (LHS of the inequality)

Let $A' = \{x \times y : x \in A\}$

Let $B_k' = \{x \times y : \underbrace{(\underbrace{y_1 \cdots y_k}_{\text{2nd space}}, \underbrace{x_{k+1} \cdots x_n}_{\text{1st space}})}_{n \text{ coordinates}} \in B\}$ Composite vector $z_k(x,y)$

We will show

1) $\mathbb{P}(A \circ B) = \mathbb{P}_{12}(A' \circ B_0')$

$B_n' = \{x \times y : (y_1 \cdots y_n) \in B\} = B''$

A and B are increasing events.

2) $\mathbb{P}_{12}(A' \circ B_{k-1}') \leq \mathbb{P}_{12}(A' \circ B_k') \rightarrow$ as k increases this prob. increases as well

$\mathbb{P}(A \circ B) = \mathbb{P}_{12}(A' \circ B_0') \leq \cdots \leq \mathbb{P}_{12}(A' \circ B_n') = \mathbb{P}(A)\mathbb{P}(B)$

3) $\mathbb{P}_{12}(A' \circ B_n') = \mathbb{P}(A)\mathbb{P}(B)$ You can check this.

2) is the hard part. We will show that (Inductive step) there is an injection from

$A' \circ B_{k-1}' \longrightarrow A' \circ B_k'$

will construct a map, and show that it's an injection.

This will show $\mathbb{P}_{12}(A' \circ B_{k-1}') \leq \mathbb{P}_{12}(A' \circ B_k')$

$(x_1, \dots 0, \dots, x_n) \in A$, but $(x_1, \dots, 1, \dots x_n) \notin A$

$A' = \{x \times y : x \in A\}$

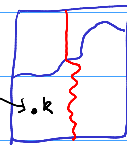
$B'_{k-1} = \{x \times y : (y_1 \cdots y_{k-1}, x_k, x_{k+1}, \dots, x_n) \in B\}$

I have already REPLACED $k-1$ coordinates.

We will split up $A' \circ B'_{k-1} = C_1 \cup C_2$

$C_1 = \{ x \times y : A' \circ B'_{k-1}$ happens for both $x_k = 0$ or $1 \}$

$\Rightarrow (y_1 \cdots y_{k-1}, y_k, x_{k+1}, \cdots x_n) \in B$



$C_2 = \{x \times y : x_k = 1$ and $A' \circ B'_{k-1}$ occurs but if $x_k = 0$ $A' \circ B'_{k-1}$ would not occur$\}$. (x_k is PIVOTAL to the occurrence of $A \circ B$)

$C_3 = \{x \times y : x_k = 0$ and $A' \circ B'_{k-1}$ occurs but if $x_k = 1$ $A' \circ B'_{k-1}$ would not occur$\}$.

→ This cannot be relevant in the case where A and B are increasing.

POLL

One of these sets is irrelevant for us.

C_1	C_2	C_3
A	B	C

In C_2 x_k is a 'pivotal edge for $A \circ B_{k-1}$' $x_k = 1$

In C_1 x_k is irrelevant.

OK, if x_k is pivotal, then there are 2 possibilities

C_2^A = x_k is pivotal for A' (but not for B_{k-1}') follows from disjoint occurrence

C_2^B = x_k is pivotal for B_{k-1}' (but not for A')

$$C_2^A = \{ x \times y : \exists\, I \subseteq \{1, \dots, n\} \text{ st } k \in I \quad C(x \times y, I) \subseteq A,\ C(x \times y, I^c) \subseteq B \}$$

C_2^B can be defined similarly.

Now we construct the injection φ on $A' \circ B_{k-1}'$.

On C_1, the value of x_k does not matter.

to neither A nor B.

In particular $x \times y \in A' \circ B_{k-1}'$

$\Leftrightarrow (x_1 \dots x_k, \dots x_n) \in A, \quad (y_1 \dots y_{k-1}, x_k, \dots x_n) \in B$

But the composite vector

$$(y_1 \cdots y_k, x_{k+1} \cdots x_n) \in B$$

$\Rightarrow \quad x \times y \in A' \circ B_k'$

On C_1

$\varphi(x \times y) = x \times y \in A' \circ B_k'$

(identity)

On C_2^A ← x_k matters to A

as well since x_k does not matter to B'_{k-1}, $(y_1, \ldots, y_{k-1}, x_k, \cdots x_n) \in B$

↕ x_k cannot matter to B so its obvious that

and $(y_1, \cdots y_{k-1}, y_k, \cdots\cdots, x_n) \in B$

$\varphi(x \times y) = x \times y \in A' \circ B_k'$

On C_2^B, set

→ does not matter to A

$(x_1 \cdots, x_k = 1, x_{k+1}, \cdots x_n) \in A$

$(y_1 \cdots, x_k, x_{k+1}, \cdots x_n) \in B$

→ matters to B.

$$\varphi((x_1 \cdots x_n) \times (y_1 \cdots y_n))$$

$$= \underbrace{(x_1 \cdots y_k \cdots x_n)}_{x'} \times \underbrace{(y_1 \cdots x_k \cdots y_n)}_{y'}$$ ← φ swaps x_k and y_k.

<u>Claim</u>: $x' \times y' \in A' \circ B_k'$ (kind of clever and subtle because

$\varphi(x \times y) \in A' \circ B_k'$

when $x \times y \in A' \circ B'_{k-1}$

Composite vector $z = (y_1 \cdots, y_{k-1}, 1, x_{k+1} \cdots x_n) \in B$

Composite vector $z_k' = (y_1' \cdots, y_{k-1}', y_k', x_{k+1}' \cdots x_n')$

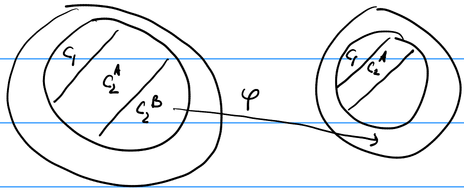
$= (y_1 \cdots, y_{k-1}, x_k, x_{k+1}, \cdots x_n) \in B.$ ← z_{k-1}

Next to show injectivity.

(Swipped this injection)

On $C_1 \cup C_2^A$ φ is the identity.

So we only need to check that on C_2^B φ does not map to $C_1 \cup C_2^A$



If $x \times y \in C_2^B$ then certainly k^{th} coordinate must be 1 for B. $x' \times y' = (x_1, \dots, y_k, \dots x_n) \times (y_1, \dots, 1, \dots y_n)$ cannot be in $C_1^{A \circ B'_{k-1}}$ since this would mean that the k^{th} coordinate is irrelevant to $A \circ B'_{k-1}$ which is not true.

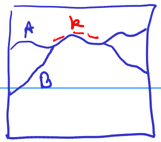
Similarly in cannot be in C_2^A which says that k^{th} coordinate is pivotal for A, which is not true by definition.

φ also preserves measure since

$$P_{12}((x_1 \cdots x_n) \times (y_1, \cdots y_n)) = \prod_{i=1}^{n} p(x_i) \prod_{j=1}^{n} p(y_j)$$

$\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B)$

A, B



$$\mathbb{P}(A \cap B \mid B) \leq P(A)$$