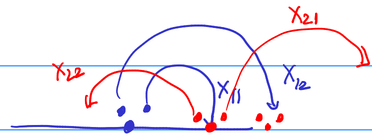
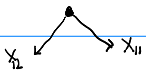
lec 12 A short return to the BRW.

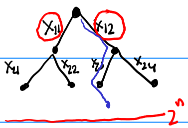


Suppose $d=1$ and have a particle at the origin.



always gives rise to 2 particles and $\{X_{ij}\} \sim N(0,1)$ independent.

X_{11} & X_{12} are iid $N(0,1)$ jumps.



and so on. Let us try to compute the scaled shape of the BRW. (locations of the 2^n particles after n steps)

displacement of $Z_r^{(n)} = \sum_{i=1}^{n} X_{ij}$ appropriate path on the tree

$\frac{Z_r^{(n)}}{n} = I_r^{(n)}$

2^n

Convex hull containing all $I_r^{(n)}$.

Biggins convex fn. $k(\theta) = \log \mathbb{E}\left[e^{-\theta X_1} + e^{-\theta X_2} \right]$ Gaussian indep.

$= \log\left[e^{\theta^2/2} + e^{\theta^2/2} \right] = \log 2 + \frac{\theta^2}{2}$

$\int e^{-\theta t} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt = e^{\theta^2/2}$

$k^*(y) = \inf_\theta \left\{ \theta y + \log 2 + \theta^2/2 \right\} = \inf_\theta \left\{ \frac{(\theta+y)^2}{2} - \frac{y^2}{2} + \log 2 \right\}$

$\frac{\theta^2}{2} + \theta y = \frac{(\theta+y)^2}{2} - \frac{y^2}{2}$

Pick $\theta = -y$ and get $k^*(y) = \log 2 - \frac{y^2}{2}$

$k^*(y) \geq 0$ whenever $|y| \leq \sqrt{2\log 2}$

convex hull of the particles in the BRW $\mathcal{P} = \{ y \mid k^*(y) \geq 0 \}$

$\max(Z_1^{(n)} \dots Z_{2^n}^{(n)})$

$\sqrt{2\log 2}$

I have another question: suppose I want to find a limiting particle density instead:

$\mu_n = \frac{1}{2^n} \sum \delta_{Z_r^{(n)}/\sqrt{n}}$ delta fn a meas.

mass $\frac{1}{2^n}$ to each of these particles.

f = max of the BRW.

$\approx \sqrt{2\log 2}\, n$

Can we compute this using Biggin's idea? $\mu_n \xrightarrow{d} \mu$?

To analyze a meas. look at its Laplace transform.

$$\mathbb{E}\int e^{\theta y}\,d\mu_n = \mathbb{E}\,\frac{1}{2^n}\sum_i \int e^{\theta y}\,\delta_{\frac{Z_i^{(n)}}{\sqrt{n}}} = \frac{1}{2^n}\,\mathbb{E}\Big[\sum_i e^{\theta\frac{Z_i^{(n)}}{\sqrt{n}}}\Big]$$

$$= \frac{\left(e^{k\left(\frac{\theta}{\sqrt{n}}\right)}\right)^n}{2^n} \approx \left(\frac{2e^{\theta^2/2n}}{2}\right)^n = e^{\theta^2/2}$$

$$\int_{-\infty}^{t} d\mu_n = \#\text{ of delta fns I encounter up to } t$$

$$= \left|\left\{ i \;\middle|\; \frac{Z_i^{(n)}}{\sqrt{n}} \leq t \right\}\right|$$

$$\mathbb{E}\left[e^{\theta\frac{Z_r^{(n)}}{\sqrt{n}}}\right] = \left[e^{k\left(\frac{\theta}{\sqrt{n}}\right)}\right]^n$$

$$\mathbb{E}\left[e^{\frac{\theta}{\sqrt{n}}X_1}\right]^n \to \left(e^{\frac{\theta^2}{2n}}\right)^n$$

Notice :

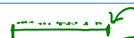
1) $\frac{Z_n}{\sqrt{n}} \leftarrow$ this was different from the $\frac{Z_n}{n}$ scaling in Biggins

2) $e^{\theta^2/2}$ is the Laplace transform of the Normal distro.

So $\mu = e^{-\frac{x^2}{2}}\,dx$. The limiting empirical measure ought to be Gaussian

$\mu_n \Rightarrow \mu$, then μ had better be a Gaussian!

$\int e^{\theta y}\,d\mu_n$ is random $\longrightarrow e^{\theta^2/2}$



$-\sqrt{2\log 2}$ $\sqrt{2\log 2}$

OK, lets see how to "find" the maximum of the BRW.

When we integrate the measure μ_n over some interval

$$X_i = \frac{Z_i^{(n)}}{\sqrt{n}}$$

$[a,\infty)$, $\displaystyle\int_a^\infty d\mu_n = \frac{1}{2^n}\sum_{i=1}^{2^n}\int_a^\infty \delta_{X_i}(u)\,du = \frac{1}{2^n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i\in[a,\infty)\}}$

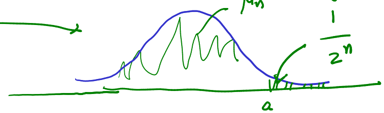
$$= \frac{\#\text{ of } X_i \text{ st } X_i\in[a,\infty)}{2^n}$$

X_{2^n} X_{2^n-1} $X_1 \leftarrow$ max of the BRW.

Thus if we order $X_{2^n} < X_{2^n-1} \cdots\cdots < X_1$ then

$$\int_{X_1}^\infty d\mu_n = \frac{1}{2^n}$$

So as a heuristic, let us suppose $\mu_n \approx \mu$ (Gaussian) assume the measure has approached its limit.



limiting density Gaussian

Then the location of the maximum X_1 is approximately a st

$$\int_a^\infty \underset{\text{limit}}{\mu} \approx \frac{1}{2^n}$$

$$= \int_a^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,du \approx e^{-\frac{a^2}{2} - \log a} = \frac{1}{2^n}$$

integration by parts — Exercise

solve for this a then I will get the location of $X_1 = \frac{Z_1^{(n)}}{\sqrt{n}}$ (max of the BRW)

which gives $\frac{a^2}{2} + \log a \approx n \log 2$

$$\Rightarrow \quad a \approx \sqrt{2\log 2}\; n + O(\log n)$$

log is very small compared to $\sqrt{n}$

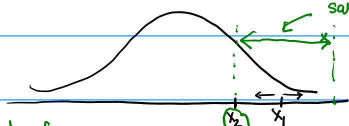
$X_1 \approx a \Rightarrow Z_1^{(n)} \approx \sqrt{2\log 2}\, n + O(\sqrt{n})$ correction

smaller order

Since we looked at $\frac{Z^{(n)}}{\sqrt{n}}$ in the empirical measure,

$$X_1 \approx \sqrt{2\log 2}\; n + O(\log n)$$

would be a research project by itself.



square root of variance is where I expect to find the maximal particle.

2nd particle should at least bound the left boundary of the maximal particle.

Apply Efron-Stein and Talagrand to Var(f) and see how close we get to the "truth".

Integrate until you get 2 particles

$$f(a_1) = \frac{a_1^2}{2} + \log a_1 \approx n\log 2$$

$$\hookrightarrow f(a_2) = \frac{a_2^2}{2} + \log a_2 \approx (n-1)\log 2$$

$$\log 2 = |f(a_1) - f(a_2)| \approx \left| \frac{a_1^2 - a_2^2}{2} + \log\frac{a_1}{a_2} \right|$$

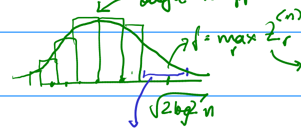
$\Rightarrow$ The "variance" of X_1 can be at most

$$\log 2 = \underbrace{\frac{|a_1 - a_2|\,|a_1 + a_2|}{2}}_{\sqrt{n}} + \log\frac{a_1}{a_2} \qquad \text{— \#3}$$

As we saw before $a_i \approx \frac{Z_i^{(n)}}{\sqrt{n}}$

Get some heuristics for the "truth".

ought to approach a gaussian



how big is the variance of the maximal particle.

location of max — location of 2nd.

from #3 plug in $a_1 = \frac{Z_1^{(n)}}{\sqrt{n}}$ and $a_2 = \frac{Z_2^{(n)}}{\sqrt{n}}$

$Z_1 \approx \sqrt{2\log 2}\, n + \sqrt{n}\log n$

$Z_2 = \sqrt{2\log}\,(n-1) + \sqrt{n-1}\log\sqrt{n-1}$

So we get $\log 2 \approx \left|\frac{Z_1 - Z_2}{\cancel{\sqrt{n}}}\right| \cancel{\sqrt{n}} + \underbrace{\log\left(\frac{Cn + \sqrt{n}\log n}{C(n-1) + \log(n-1)}\right)}_{O\left(\frac{1}{n}\right)} \to o(1)$

Thus $|Z_1 - Z_2| \approx O(1)$

Maybe this suggests that the variance should be of order $O(1)$. Let us try to estimate the variance using Efron-Stein and Talagrand L1-L2 (Gaussian version)

A way to assess the quality of our inequalities.

<u>Theorem</u>: (Gaussian-Poincare). Let $f: \mathbb{R}^n \to \mathbb{R}$

and $\mathbb{E}[g(X_1 \cdots X_n)]$

n dimensional iid Gaussian meas. on $\mathbb{R}^n$

$$= \int g(u_1 \cdots u_n) \frac{e^{\boxed{-\frac{(u_1^2 + \cdots u_n^2)}{2}}}}{(2\pi)^{n/2}} du_1 \cdots du_n$$

Then,

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \|\partial_i f\|_2^2$$

$C = 1$

A continuous version of Efron-Stein

$\mathbb{E}[(f(X) - f(X_i))^2]$

measuring influence of the ith variable.

<u>Theorem</u> (Talagrand's L1-L2 inequality for Gaussian) Under the same setup as above.

some constant indep of f and n.

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1+\log \frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}}$$ → Poincaré

In the discrete case "$\partial_i f$" = "$\rho_i f$"

locations of the BRW particles after n generations

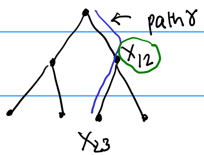
let $f = \max\left(Z_1^{(n)}, \ldots, Z_{2^n}^{(n)}\right)$

$f : \mathbb{R}^{2^{n+1}-1} \longrightarrow \mathbb{R}$ $\quad$ f is a function of the Gaussians $\{X_{ij}\}$

Gaussian

$$\partial_{ij} f = \mathbb{1}_{\{X_{ij} \text{ was in a maximal path}\}}$$

$$\mathbb{E}\left[|\partial_{ij} f|\right] = \|\partial_{ij} f\|_1 = \mathbb{P}\left(X_{ij} \text{ is in the maximal path}\right)$$



$$\|\partial_{ij} f\|_1 = \mathbb{P}\left(X_{ij} \text{ was in a max path}\right) = \frac{1}{2^i}$$

Since there are $2^{(i)}$ $\{X_{ij}\}$ variables at depth i

Guess what bound we're going to get for

$\mathrm{Var}(f) \leq$?

A	B	C	D
$O(n)$	$O(n^2)$	$\log n$	$O(1)$

In fact $\|\partial_{ij} f\|_2^2 = \|\partial_{ij} f\|_1 = \frac{1}{2^i}$

$$\mathbb{E}\left[|\partial_{ij} f|^2\right] = \mathbb{E}\left[\mathbb{1}_{\{X_{ij} \text{ is in a max path}\}}^2\right] = \mathbb{E}\left[\mathbb{1}_{\{X_{ij} \text{ is in a max path}\}}\right]$$

Efron-Stein gives

Thus $\mathrm{Var}(f) \leq \overbrace{\sum_{i=1}^{n}}^{n \text{ gens}} \overbrace{\sum_{j=1}^{2^i}}^{\text{at each level}} \frac{1}{2^i} = \sum_{i=1}^{n} 1 = n$

$$f \approx \overbrace{\sqrt{2\log 2}\, n + \sqrt{n}\log n}^{\mathbb{E}f} + \boxed{O(1)} \text{ corrections}$$

$O(\sqrt{n})$ corrections

Talagrand gives

$$\mathrm{Var}(f) \le \sum_{i=1}^{n} \sum_{j=1}^{2^i} \frac{\frac{1}{2^i}}{\log \frac{\|\partial_{ij} f\|_2}{\|\partial_{ij} f\|_1}}$$

$$\mathrm{Var}\Big(\frac{\overset{\text{random}}{f} - \overset{\text{deterministic}}{\mathbb{E} f}}{\underset{\text{scaling}}{n}}\Big) = \frac{1}{n^2}\, n \longrightarrow 0$$

"At least a law of large numbers should hold"

You can prove this with conv. in probability.

POLL

What is this estimate here?

A	B	C	D
n	$\sqrt{n}$	$\log n$	$O(1)$

$$\|\partial_{ij} f\|_2^2 = \|\partial_{ij} f\|_1 = \frac{1}{2^i} \Rightarrow \frac{\|\partial_{ij} f\|_2}{\|\partial_{ij} f\|_1} = 2^{i/2}$$

$$\mathrm{Var}(f) \le \sum_{i=1}^{n} \sum_{j=1}^{2^i} \frac{1}{2^i} \frac{1}{\log 2^{i/2}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{2^i} \frac{1}{2^i} \cdot \frac{1}{i/2} = \sum_{i=1}^{n} \frac{2}{i} \approx O(\log n)$$

Because the right order is $O(1)$.

$$f \approx O(n) + \underbrace{\xi}_{\text{(Analog of limiting random fluctuations)}} \mathrm{Var}(1)$$

"Randomly shifted Gumbel distribution"

Next: I want to prove that $\mathrm{Var}(T(0,x))$ is at least $O(\log|x|_1)$. This is a result of Newman and Piza. But before I can prove this, I need a few fundamental results.

Theorem (Kesten, Aspects) Assume $F(0) < p_c$ (zero does not percolate) $\exists\ \epsilon, C$ st

$$\mathbb{P}(\exists \text{ a path } \Gamma \text{ containing } 0 \text{ with } |\Gamma| \geq n \text{ but } T(\Gamma) < \epsilon n) \leq e^{-Cn}$$

] exponentially unlikely.

] Implies that the geodesic cannot be too long.

This requires a new set of correlation inequalities.

First is classical percolation inequality due to Harris.

The second is a lovely inequality due to van den Berg and Kesten. It is a "separated occurence inequality" and

Return to FPP.

$$C_1 \leq \mathrm{Var}(T(0,x)) \leq \frac{C|x|_1}{\log|x|_1}$$

Kesten 1990s

2000s (BKS)

1995 (Newman-Piza)

$\mathrm{Var}(T(0,x)) \geq C\log|x|_1$ "divergence of the variance"

Requires several new analytic ingredients.

1) Kesten's estimate for the length of the geodesic

↳ BK inequality.

BK conjecture

↳ BKR inequality.

2) "discrete Fourier analysis" on the discrete hypercube. $\{0,1\}^n$.

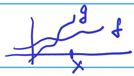
I will spend some time proving it.

Harris-FKG inequality

Used in statistical mechanics & the Ising model

In $\mathbb{R}^1$. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be nondecreasing functions. Let X be an r.v.

Harris ineq.



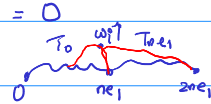
Then $\mathrm{Cov}(f(X), g(X)) = \mathbb{E}[fg] - \mathbb{E}[f]\mathbb{E}[g]$
≥ 0

In other words f and g are positively correlated.
This seems kind of obvious right? When X is large, so is f and so is g.

$\mathbb{E}[(f-Ef)(g-Eg)]$
$=: \mathrm{Cov}(f,g)$

Co-variance (positive) means f and g vary together.

If f and g are indep.
$= \mathbb{E}[f-Ef]\,\mathbb{E}[g-\mathbb{E}g]$
$= 0$



$\mathrm{Cov}(T_0, T_{ne_1}) \geq 0$

Pf: Let X, X' be indep. copies. (of the same rv)

$\mathbb{E}[(f(X)-f(X'))(g(X)-g(X'))]$

$= \mathbb{E}[f(X)g(X) - f(X')g(X) - f(X)g(X') + f(X')g(X')]$

$\mathbb{E}[f(X')g(X)] = E[f(X)]\mathbb{E}[g(X')]$
$= \mathbb{E}[f(X)]\mathbb{E}[g(X)]$

$= 2\left(\mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)]\mathbb{E}[g(X')]\right)$

$= 2\,\mathrm{Cov}(f(X), g(X))$

$$\mathbb{E}\Big[\overbrace{\big(f(X)-f(X')\big)\big(g(X)-g(X')\big)}^{\text{always positive}}\Big]$$

Suppose $X > X'$. Then $f(X)-f(X') \geq 0$ (monotonicity)

and $g(X)-g(X') \geq 0$

Similarly for $X \leq X'$ $\quad f(X)-f(X') \leq 0 \quad g(X)-g(X') \leq 0$

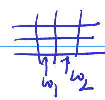
$$\big(f(X)-f(X')\big)\big(g(X)-g(X')\big) \geq 0$$

and so $\operatorname{cov}(f(X), g(X)) \geq 0$.

This can be extended to $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}$ where f, g are nondecreasing in each coordinate. You do this by conditioning and is left as an exercise.

$f = T(0, x, \omega_1, \omega_2, \dots)$

f is a monotone fn in each of the ω_i



Ex: $T(x, x+ne_1) = f$

$T(y, y+ne_1) = g$

$\operatorname{cov}(f, g) \geq 0$

An event A is called increasing if $1_A(X_1 \dots X_n)$ is nondecreasing in each coordinate.

Harris' inequality is applied to increasing (or decreasing events) and stated as $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\,\mathbb{P}(B)$.

$$\mathbb{E}[1_A 1_B] \geq \mathbb{E}[1_A]\,\mathbb{E}[1_B]$$

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\,\mathbb{P}(B)$$

lower bound

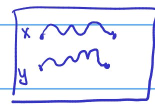
also need some "matching" upper bound.

How about going the other way? Can we say $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\,\mathbb{P}(B)$? If A is increasing and B is decreasing, yes.

c

$T(x, x+ne_1) = T_x$

$\rightarrow T_y$



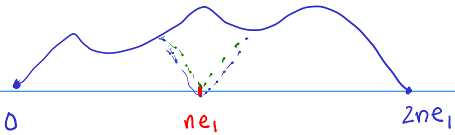
$\mathbb{P}(T_x > a, T_y > b) \geq \mathbb{P}(T_x > a)\,\mathbb{P}(T_y > b)$

no matching upper bound.

$\mathbb{P}(T_x > a, T_y > b$ disjointly but their geodesics DO NOT INTERSECT$)$

then $\leq \mathbb{P}(T_x > a, T_y > b)$

Berg and Kesten developed a remarkable inequality with a new operation $\underbrace{A \circ B}$ ← disjoint occurrence that replaces $A \cap B$. This is extremely useful in percolation and has a cool "rags to riches" story.



$\mathbb{P}(\gamma_{0\,2ne_1}$ passes through $ne_1) \longrightarrow 0$?

BKS (2003)