Lec 11 (logarithmic Improvement)

$\mathrm{Var}(T(0,x)) \leq C|x|_1$ (Kesten)

Efron-Stein inequality.

$$\mathrm{Var}(T(0,x)) \leq \frac{C|x|_1}{\log|x|_1}$$

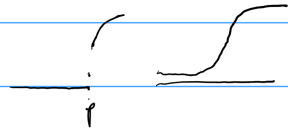
$\mathrm{Var}(T(0,x)) = |x|^{2/3}$ (Expect from simulations & solvable models)

1988 Kahn-Kalai and Linial. They were studying Boolean fns $f: \{0,1\}^n \to \mathbb{R}$

Measure how much fns f "vary" when you flip individual bits.

1994 (M. Talagrand) $\to$ A generalization of KKL ineq. (P. IHES). (Russo's approx. 0-1 law)

Ann. of Prob.



Lec 11 (logarithmic Improvement)

Two tools used in previous:

1) $\underline{GEO}(0,x) = \bigcap \{\text{geodesics from } 0 \text{ to } x\}$ $\quad |GEO(0,x)| \le C|x|_1$

2) $T_i(0,x) - T(0,x) \le \tau'_{e(i)} \mathbb{1}_{\{e(i) \in GEO(0,x)\}}$ $\quad$ (INFLUENCE)

$\uparrow$ was $T(0,x)$ with i^{th} edge replaced by an independent copy.

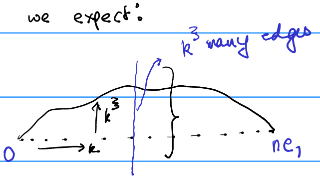
Sum over i always get

$$\mathrm{Var}(T(0,x)) \le C\,\mathbb{E}[GEO(0,x)] \simeq O(\|x\|_1)$$

In $d=1$ each edge $(i,i+1)$ has a high influence since it must be taken on a path from 0 to ne_1

In higher dim $d \geq 2$, each edge has lower influence,

In $d \geq 2$ we expect:



$\xi = \frac{2}{3}$

each edge equally likely, then since there are Approximately $(n^\xi)^{2(d-1)}$ edges, for bounded weights

$$\mathbb{E}[(T_i(0,x) - T(0,x))^2] \le C\,\mathbb{P}(i \in GEO(0,x)) \text{ if } i \text{ has 1st coordinate } k.$$

This makes little difference to the Efron-Stein ineq.

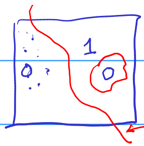
2003 Benjamini-Kalai-$\boxed{\text{Schramm}}$ applied

Talagrand's ineq. to get $\mathrm{Var}(T(0,x)) \le \frac{C|x|_1}{\log|x|_1}$

weights $\tau_{e_i} \sim \mathrm{Bernoulli}(1/2)$

(Benaïm Rossignol, Damron-Hanson-Sosoe)

→ Originally a complex analyst who got intr in prob.



lots of amazing conjectures about the behavior at criticality.

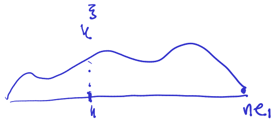
(Aizenmann, Cardy, etc.)

Loewner evolution. Schramm had this idea to drive this Loewner evolution by Brownian motion (SLE (Schramm-Loewner evolution))

Percolation of the triangular lattice. S. Smirnov. (2000s) – Fields Medal.

(Juan has Inventiones paper with Smirnov)

(Schramm had just barely crossed 40).



$k = 1, \dots, n$

$$\sum_{\text{edges } i} \mathbb{P}(i \in GEO(0,x)) \approx \sum_{k=1}^{n} \frac{1}{k^{\xi}} \cdot k^{\xi} \approx O(n)$$

We do not get an improvement using Efron-Stein even if we use the best "available" estimates for the influence.

Improvement from $\frac{|x|}{\log|x|}$ to $|x|^{1-\epsilon}$ for any $\epsilon > 0$

is a BIG OPEN PROBLEM (S. Chatterjee told that J. Bourgain told him that this is "huge" open problem)

So to take all this new information into account, we need a BETTER INEQUALITY.

The first improvements came in a Computer Science work called "KKL" (Kahn-Kalai-Lineal) in the form of a boolean influence inequality. 1988

Talagrand had a series of incredible papers that generalized their inequality in the 1994 (Russo's approximate 0-1 law)

→ Conc. of Meas and Isoperimetric ineq. in product spaces (Pub IHES) 1997

The 1st proof of sublinear variance was due to Benjamini-Kalai-Schramm (2003), and uses Talagrand's "influence ineq. on a hypercube", and thus was restricted to $\{a,b\}$ valued rvs.

Schramm was an interesting guy.

Since 2003, the result for FPP has been generalized to arbitrary weights, BUT THE INEQUALITY HAS NOT BEEN IMPROVED.

Talagrand's inequality (L1-L2)

$Z_e \in \{a,b\}$ $\quad \mathbb{P}(Z_e = a) = \frac{1}{2}$ $\quad 0 < a < b < \infty$.

for any $f : \Omega \longrightarrow \mathbb{R}$, $\Omega = \{a,b\}^n$ (Boolean Hypercube)

$\omega \in \Omega$ $\quad \hat{\omega}_j = \omega$ with j coordinate replaced with the opposite value (replacing by an independent copy.) Let $\rho_j f = \frac{1}{2}\left[f(\omega) - f(\hat{\omega}_j)\right]$

Measuring the influence of the j^{th} coordinate.

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \frac{\|\rho_j f\|_2^2}{1 + \log \frac{\|\rho_j f\|_2}{\|\rho_j f\|_1}}$$

C: doesn't depend on n

$\|\rho_j f\|_2^2$: this will become the $\mathbb{E}\left[\mathbb{1}_{\{j \in GEO\}}\right]$

If we could show that $\frac{\|\rho_j f\|_2}{\|\rho_j f\|_1} \geq n^\alpha$ then we will end up with the $\log n$ improvement we want.

Again let's start with some examples.

Ex: $f(\omega) = \sum_{i=1}^{n} \omega_i$ $\quad |\rho_j f| = \frac{1}{2}(b-a) = C_2$

$$|\rho_j f(\omega)| = \frac{1}{2}\left[\sum_{i=1}^{n} \omega_i - \sum_{\substack{i=1 \\ i \neq j}}^{n} \omega_i - (\omega_j)^c\right] = \frac{1}{2}\left|(\omega_j - \omega_j^c)\right| \leq \frac{b-a}{2}$$

So $\frac{\|\rho_j f\|_2}{\|\rho_j f\|_1} = 1$ So no improvement to the variance bnd.

$$\mathrm{Var}(f) \leq C \sum_{i=1}^{n} \frac{C_2}{1 + \log 1} = Cn$$

(This is the right order)

POLL: $\omega \in \{a,b\}^n$ with uniform meas. and claiming that $\{\omega_j\}_{j=1}^n$ are iid random variables

YES OR NO

Ex: $(\Omega = \{a,b\}^n)$ with uniform meas. Then $\{\omega_j\}_{j=1}^n$ are iid rvs with

$\mathbb{P}(\omega_j = a) = \mathbb{P}(\omega_j = b) = \frac{1}{2}$.

$$\underbrace{\mathbb{P}\left((\omega_{i_1}, \dots \omega_{i_k}) = (x_1, \dots, x_k)\right)}_{\text{LHS is also } \frac{1}{2^k}} \overset{\uparrow}{=} \prod_{j=1}^{k} \mathbb{P}(\omega_{i_j} = x_j) = \frac{1}{2^k}$$

definition of independendence.

Efron-Stein $\mathrm{Var}(f) \leq \overset{C}{\cancel{\tfrac{1}{\log n}}}$ (for the Gaussian distribution)

$\mathrm{Var}(f) \hat{\approx} \frac{1}{\log n}$

$a < b$

Ex: $f(\omega) = \max\{\omega_1, \dots, \omega_n\}$

This problem is trivial on the hypercube.

$P(f = b) = 1 - \frac{1}{2^n}$ $\quad P(f = a) = \frac{1}{2^n}$ $\quad E[f] = b\left(1 - \frac{1}{2^n}\right) + a \overbrace{\frac{1}{2^n}}^{p_n}$

$E[f] = (b-a)E[X] + a$
$= (b-a)(1-p_n) + a$

$f = (b-a)X + a \qquad X \sim \text{Bernoulli}(1-p_n) \qquad p_n = \frac{1}{2^n}$

$P(X = 1) = 1 - p_n$
$P(X = 0) = p_n$

We know $\mathrm{Var}(f) = (b-a)^2 p_n(1-p_n) \approx \Theta\left(\frac{1}{2^n}\right) \to$ #1

$E[f] \to b$ as $n \to \infty$

$\mathrm{Var}(f) = (b-a)^2 \mathrm{Var}(X)$
$= (b-a)^2 p_n(1-p_n)$

Efron-Stein: $\mathbb{E}[(f - f_i)^2]$. You can only induce a change in f when $\omega_j' = b$ and $\omega_i = a$ for $i \neq j$ OR if $\omega_i = a \ \forall i$.

$\mathrm{Var}(f) \leq \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}[(f - f_i)^2]$

$P(\omega_i = a \ \forall i \neq j) = \frac{1}{2^{n-1}}$

If $\omega_j' \neq \omega_j$ Then $(f_i - f)^2 = (b-a)^2$

ω_j' needs to be independent of ω_j.

$\mathbb{E}[(f - f_i)^2] = (b-a)^2 \frac{1}{2^{n-1}} \cdot \left(\frac{1}{2}\right)$

$\mathrm{Var}(f) \leq \frac{1}{2}\sum_{i=n}(b-a)^2 \frac{1}{2^n} = (b-a)^2 \frac{n}{2^{n+1}} \gg c\frac{1}{2^n}$

Efron-Stein gives the wrong ORDER. (It sucks just a little bit even in this simple cases)

$$\mathrm{Var}(f) \le c \sum_{i=1}^{n} \frac{\|\rho_i f\|_2^2}{1 + \log \frac{\|\rho_i f\|_2}{\|\rho_i f\|_1}}$$

Let's see what Talagrand tells us:

$$\rho_j f = \frac{1}{2}\Big[\max\{\omega_1 \dots \omega_j, \dots, \omega_n\} - \max\{\omega_1 \dots \omega_j^c, \dots, \omega_n\}\Big]$$

opposite ω_j

$$= \mathbb{1}_{\{\omega_1=a, \omega_2=a \dots, \omega_n=a\}} \mathbb{1}_{\{\omega_j = b\}} \frac{1}{2}(b-a) + \mathbb{1}_{\{\omega_1=a, \dots, \omega_n=a\}} \mathbb{1}_{\{\omega_j=a\}} \frac{1}{2}(a-b)$$

event had small prob.

Exercise:

$$E(\rho_j f)^2 = \frac{1}{4}(b-a)^2 \frac{1}{2^{n-1}}$$

$$\mathbb{E}[(\rho_j f)^2] = c_1^2 \frac{1}{2^{n-1}}$$

$$\|\rho_j f\|_2 = c_1 \frac{1}{2^{\frac{n-1}{2}}} \qquad \|\rho_j f\|_1 = c_1 \frac{1}{2^n}$$

Exercise.

$$\frac{\|\rho_j f\|_2}{\|\rho_j f\|_1} = \frac{c_1 \left(\frac{1}{2}\right)^{\frac{n-1}{2}}}{c_1 \left(\frac{1}{2}\right)^n} = 2^{\frac{n-1}{2}}$$

log will give an $O(n)$ correction

$$\Rightarrow \mathrm{Var}(f) \le c \sum_{i=1}^{n} \frac{c_1^2 \frac{1}{2^{n-1}}}{1 + \log 2^{n/2}} = c_3 \frac{1}{2^{n-1}}$$

#1b

See that from #1a and #1b, we have the correct order.

So it gets it down to the right order!

Remark: We got an improvement since $\rho_j f \approx c \mathbb{1}_{A_j}$ where A_j had low probability

You would want something similar in FPP. But BKS were not able to show that.

Lemma: $\exists$ some $\alpha > 0$ st if $|f| < C_1$ (If f is bounded by some constant)

Then $\dfrac{\|f\|_2}{\|f\|_1} \geqslant C_2 \dfrac{1}{\|f\|_1^{\alpha}}$

exponent will depend on the constant bounding $|f|$.

Hint: Use Hölder's inequality. ← Exercise to try at home.

POLL: Do you know this inequality?

YES OR NO

just bound this from above

possibly different constant but depend on n.

And thus $\mathrm{Var}(f) \leqslant C \sum_j \dfrac{\|\rho_j f\|_1}{1 + \log \frac{1}{\|\rho_j f\|_1}}$ ← $\|\rho_j f\|_2^2$ — #2

Hölder's inequality works when $|\rho_j f| < C$

f will be the passage time. $|\rho_j f| < (b-a)$

Lemma: If $|f| \leqslant C_1$ then $\|f\|_2^2 \leqslant C_1 \|f\|_1$

Pf:

$$\left|\frac{f}{C_1}\right| \leqslant 1 \qquad \|f\|_2^2 = \mathbb{E}[f^2] = C_1^2\, \mathbb{E}\left[\left(\frac{f}{C_1}\right)^2\right]$$

$$\leqslant C_1^2\, \mathbb{E}\left[\frac{|f|}{C_1}\right] = C_1 \|f\|_1$$

Now, $f = T(0,x)$ and estimate $\|\rho_j f\|_1 = \|\rho_j T(0,x)\|_1$

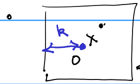
$\rho_j T(0,x) = \frac{1}{2}\left(T(0,x,\omega_1,\dots,\omega_n) - T(0,x,\omega_1,\dots,\hat{\omega}_j,\dots,\omega_n)\right)$ "Influence of the j^{th} variable"

BKS - randomization

Kalai Schramm

Define $\tilde{T}(0,ne_1)$ to approximate $T(0,ne_1)$ $x=ne_1$ (we will drop the argument of this function.

$$\begin{array}{l} T(0,ne_1) = T \\ T(x,x+ne_1) = T_x \end{array}$$ Shorthand



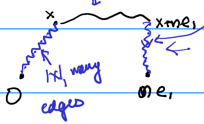
$x+ne_1$

Box of size $2k+1 \times 2k+1$ (B)

$$\tilde{T} = \frac{1}{|B|}\sum_{x\in B} T(x,x+ne_1)$$

T_x

averaging passage times over the box.



$T \leq T_x + 2b|x|_1$ $\qquad T_x \leq T + 2b|x|_1$ $\qquad T \leq T_x + 2b|x|_1$

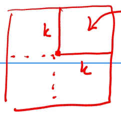
$\Rightarrow \quad |T-T_x| \leq 2b\,|x|_1$ by repeating the argument.

$|B| = (2k+1)^2 = O(k^2)$

$$|T-\tilde{T}| \leq \frac{1}{|B|}\sum_{x\in B}|T-T_x| \leq \frac{2b}{|B|}\sum_{x\in B}|x|_1 \leq \frac{2bC}{k^2}\cdot k^3$$

$\hookrightarrow k^2$

$= O(k)$



estimate this by an integral.

$$4\int_0^k\int_0^k (x+y)\,dx\,dy = O(k^3)$$

$$\int \frac{x^2}{2}\Big|_0^k dy + \int \frac{y^2}{2}\Big|_0^k dx$$

$$= \frac{k^2}{2}\int_0^k dy + \frac{k^2}{2}\int_0^k dx$$

$$|T-\tilde{T}| = O(k)$$

$$\Rightarrow \mathbb{E}\left[|T-\tilde{T}|^2\right] = O(k^2) \qquad \text{(translation invariance)}$$

$$\mathbb{E}[T] \overset{?}{=} \mathbb{E}[\tilde{T}] = \frac{1}{|B|}\sum_{x\in B}\mathbb{E}[T_x] = \frac{1}{|B|}\sum_{x\in B}\mathbb{E}[T]$$

$$\sum_{x\in B}|x|_1 = \sum_{j=1}^{2k} j\,(\#\text{ of } x \text{ with } |x|_1=j) \leq C\sum_{j=1}^{2k} j^2 \leq Ck^3$$

add and subtract $\tilde{T}$

So $\quad \mathrm{Var}(T) = \mathbb{E}\left[(T-\mathbb{E}T)^2\right]$

$\mathbb{E}\tilde{T}$

$$\leq \mathbb{E}\left[(T-\tilde{T})^2\right] + \mathbb{E}\left[(\tilde{T}-\mathbb{E}T)^2\right]$$

$$\leq Ck^2 + \mathrm{Var}(\tilde{T}) \quad \text{and thus we can}$$

bound $\mathrm{Var}(\tilde{T})$

Talagrand to $\tilde{T}$

$|\mathrm{Var}(\tilde{T}) - \mathrm{Var}(T)|$

$$\mathbb{E}[T(x,x+ne_1)] = \mathbb{E}[T_x]$$
$$\shortparallel$$
$$\mathbb{E}[T(0,ne_1)]$$

$\rho_j T_x(\text{when } \omega_j = a) = \frac{1}{2}\left(T_x(\omega_j = a) - T_x(\omega_j = b)\right)$

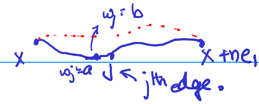
$= -\rho_j T(\text{when } \omega_j = b$ and all other coordinates fixed)

change in passage time

$\rho_j \tilde{T} = \frac{1}{|B|}\sum_{x\in B} \rho_j T_x = \frac{1}{|B|}\sum_x 1_{\{j \in GEO_x\}}$ (change)

Think about this.

$\|\rho_j \tilde{T}\|_1$ bound from above and plug into Talagrand



The passage time T_x changes when $\omega_j = a$ ONLY when ω_j is in all the geodesics from x to x+ne₁

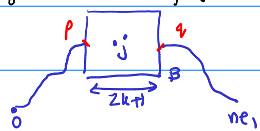
GEO_x = edges common to all geodesics from 0 to x

$\frac{b-a}{2}$

$\mathbb{E}|\rho_j \tilde{T}| \leq \frac{C}{|B|}\sum_{x\in B} \mathbb{E}\left[1_{j\in GEO_x}\right] = \frac{C}{|B|}\sum_{x\in B}\mathbb{E}\left[1_{j-x\in GEO_0}\right]$

$= \frac{C}{|B|}\mathbb{E}[\#$ edges crossed by GEO_0 in box around j of size $k]$

$\leq \frac{C}{k^2} k = \frac{C_1}{k}$ is a parameter we choose



$\mathbb{E}\left[1_{j\in GEO_x}\right]$

$= \mathbb{P}(j \in GEO_x$ from x to x+ne₁$)$

$= \mathbb{P}(j - x \in GEO$ from 0 to ne₁$)$

translation inv. of measure.

$= \mathbb{E}\left[1_{\{j-x\in GEO_0\}}\right]$

$\text{Var}(f) \leq C \sum_j \frac{\|\rho_j f\|_1}{1 + \log \frac{1}{\|\rho_j f\|_1}} \to \geq \log k$

$\text{Var}(\tilde{f}) \leq C \sum_j \frac{\|\rho_j \tilde{T}\|_1}{1 + \log\frac{k}{C_1}} \leq$

maybe different

$C \sum_j \frac{\mathbb{E}\left[\frac{1}{|B|}\sum_x |\rho_j T_x|\right]}{\log k}$

$\leq C \frac{\frac{1}{|B|}\sum_j \sum_x \mathbb{E}\left[1_{j\in GEO_x}\right]}{\log k}$

$\leq C \frac{\frac{1}{|B|}\sum_{x\in B}\mathbb{E}[\#\text{ of edges in } GEO_x]}{\log k}$

GEO_x is the geodesic from x to x+ne₁

$O(n) \leq \frac{b}{a} n$

$$\mathrm{Var}(f) \leq C \sum_j \frac{\|\rho_j \tilde{T}\|_1}{1+\log\frac{k}{C_1}} \leq \frac{C}{\log k} \sum_j \frac{1}{|B|}\sum_{x\in B} \mathbb{E}\left[\mathbb{1}_{\{j\in GEO_x\}}\right]$$

$$= \frac{C}{\log k} \frac{1}{|B|} \sum_{x\in B} \mathbb{E}\left[\sum_j \mathbb{1}_{\{j\in GEO_x\}}\right]$$

$$= \frac{C}{\log k} \mathbb{E}\left[|GEO_0|\right] \leq \frac{C}{\log k} n$$

$\log k$ factor gained.

Thus $\mathrm{Var}(f) \leq \frac{C}{\log k} n + C' k^2$ ← optimize over k

and we choose $k = \sqrt{\frac{n}{\log n}}$ to get

$$\frac{Cn}{\log n - \log\log n} + \frac{C'n}{\log n} \leq C'' \frac{n}{\log n}$$

$$\mathrm{Var}(f) \leq \frac{C n}{\log n}$$

Efron-Stein (Kesten) $\mathrm{Var}(f) \leq Cn$

Talagrand (BKS) $\mathrm{Var}(f) \leq \frac{Cn}{\log n}$

"Celebrated" result.

The constant has changed from line to line.

It seems clear that this is a reasonable estimate on the influence of an edge, and the inequality itself is somehow lacking.

$\| \rho_j \circ \hat{T} \|_1 \leq \frac{1}{k}$ is a pretty good estimate

If I were able to get

$$\mathrm{Var}(f) \leq \underbrace{\frac{Cn}{k}}_{\text{inequality}} + \underbrace{C'k^2}_{\text{averaging error}}$$

then I could choose $k = n^{1/3}$ to get the right bound.

Could a "better" inequality achieve a bound like this?

But what could produce a such an improvement in the inequality? This remains to be seen.

Some miscellaneous computations (ignore)

What to do about the L^2 norm? Let $f = T_X$

$\{\rho_j f < 0\} \Rightarrow f(\omega) < f(\hat{\omega}_j)$ so ω_j must have been its lower value. $\Rightarrow \{\rho_j f \neq 0, \ \omega_j = a\}$

Conversely, if $\{\rho_j f \neq 0$ and $\omega_j = a\} \Rightarrow \{\rho_j f < 0\}$.

But $\rho_j f \neq 0$ is independent of the value of ω_j in the following sense:

$$\rho_j f(\omega_1 \ldots, a, \ldots) = \frac{1}{2}\left[f(\omega_1 \ldots, a, \ldots) - f(\omega_1 \ldots, b, \ldots)\right]$$

$$= -\frac{1}{2}\rho_j f(\omega_1, \ldots, b, \ldots)$$

$\Rightarrow \{\omega : \rho_j f \neq 0\}$ and $\{\omega : \omega_j = a\}$ are independent.

Thus

$$\{\omega : \rho_j f < 0\} = \{\rho_j f \neq 0, \ \omega_j = a\}$$

$$\mathbb{P}(\rho_j f < 0) = \mathbb{P}(\rho_j f \neq 0)\,\mathbb{P}(\omega_j = a) = \frac{1}{2}\mathbb{P}(\rho_j f \neq 0)$$

#1

Then $\mathbb{E}[|\rho_j f|] = \mathbb{E}\left[|\rho_j f| \, \mathbb{1}_{\{\rho_j f \neq 0\}}\right]$

$$\leq \mathbb{E}[(\rho_j f)^2]^{1/2} \, \mathbb{P}(\rho_j f \neq 0)^{1/2} \underset{\text{using } (\#2)}{=} \sqrt{2}\, \|\rho_j f\|_2 \, \mathbb{P}(\rho_j f < 0) \quad (\#3)$$

Now, we need to relate

$$\{\rho_j f < 0\} = \{\rho_j f < 0,\ j \in GEO(x, x+ne_1),\ \omega_j^\circ = a\}$$

$$\subseteq \{j \in GEO(x, x+ne_1)\}$$

Thus (#3) becomes

$$\leq \sqrt{2}\, \|\rho_j f\|_2 \, \mathbb{P}(j \in GEO_x \quad)^{1/2}$$

I am not sure this helps us very much.

<u>To give as exercise</u>

Lets assume that $|\rho_e f| \leq 1$. Then, using Hölder

$$\|\rho_e f\|_2^2 = \mathbb{E}[(\rho_e f)^2]$$

$$\mathbb{E}[(\rho_e f)^{\alpha+\beta}] \leq \mathbb{E}[(\rho_e f)^{\alpha p}]^{1/p} \, \mathbb{E}[(\rho_e f)^{\beta q}]^{1/q}$$

$\alpha = 1/2 \quad p = 4 \quad q = 4/3 \qquad (\rho_e f)^{2/3} \leq (\rho_e f)^1$

$$\|\rho_e f\|_1^{1/4} \leq \|\rho_e f\|_2^{1/2}$$

$$\frac{\|\rho_{ef}\|_2}{\|\rho_{ef}\|} \geq \frac{1}{\|\rho_{ef}\|_1^{1/2}}$$

Thus $$\log \frac{\|\rho_{ef}\|_2}{\|\rho_{ef}\|} \geq \frac{1}{2} \log \frac{1}{\|\rho_{ef}\|_1}$$

This is a pretty clever trick!