Lec10 : Continuing Efron-Stein

$Var(f) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\left[(f-f_i)^2\right]$

Pf: $Z = f(X_1, X_2, \ldots X_n)$

$Z_i = f(X_1, X_2, \ldots X_i', \ldots X_n)$ indep copy.

$\mathbb{E}[g \mid \mathcal{F}_i] = \mathbb{E}[g \mid X_1, \ldots X_i]$ conditional expectation

$= \int g(\underbrace{X_1, \ldots, X_i}_{\text{fixed \#s}}, \underbrace{X_{i+1}, \ldots, X_n}_{\text{averaging over}}) \, dF(x_{i+1}) \ldots dF(x_n)$

$\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$

Conditional expectation is a function of $X_1, \ldots, X_i$

$\mathbb{E}[g \mid \mathcal{F}_i]$ $\quad \mathcal{F}_i = \sigma(X_1, \ldots, X_n)$

What is $\mathbb{E}[Z \mid \mathcal{F}_i]$ called?

$\mathbb{E}[Z \mid \mathcal{F}_i]$ is a random variable depending on $X_1, \ldots, X_i$

POLL.

$\mathbb{E}\left[\mathbb{E}[Z \mid \mathcal{F}_i] \mid \mathcal{F}_{i-1}\right] = \mathbb{E}[Z \mid \mathcal{F}_{i-1}]$

Doob Martingale of a fn of n random variables.

$\mathbb{E}[Z \mid \mathcal{F}_n] = Z$ (if you freeze all variables)

$\mathbb{E}[Z \mid \mathcal{F}_0] = \mathbb{E}[Z]$

$$\mathbb{E}\left[(Z-\mathbb{E}Z)^2\right] = \mathrm{Var}(Z) = \mathrm{Var}(f)$$

Common decomposition of the Doob Martingale.

$$Z - \mathbb{E}Z = \sum_{i=1}^{n} \underbrace{\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_{i-1}]}_{\text{martingale increments. } \Delta_i} = \sum_{i=1}^{n} \Delta_i$$

$\mathbb{E}[\Delta_i \Delta_j]$ $(i<j)$

$$= \mathbb{E}\Big[\big(\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_{i-1}]\big)\big(\mathbb{E}[Z \mid \mathcal{F}_j] - \mathbb{E}[Z \mid \mathcal{F}_{j-1}]\big)\Big]$$

$X_1 \ldots X_i$; $X_1 \ldots X_{i-1}$

like a constant

tower property

$$= \mathbb{E}\big[\mathbb{E}[\Delta_i \Delta_j \mid \mathcal{F}_i]\big]$$

$i<j$

$$= \mathbb{E}\Big[\Delta_i\Big(\mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{F}_j] \mid \mathcal{F}_i\big] - \mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_i\big]\Big)\Big]$$

$\mathbb{E}[Z \mid \mathcal{F}_i]$

$$= \mathbb{E}\Big[\Delta_i\big(\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_i]\big)\Big]$$

$= 0$

POLL: Have you have seen this tower property?

A = YES

B = NO

Δ_i is a function of $X_1 \ldots X_{i-1}$

Δ_i is $\mathcal{F}_i$ measurable.

$X_1 \ldots X_i \ldots X_j$

$X_1 \ldots X_i \ldots X_j$

integrating over them

Martingale differences

So $\boxed{\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2]}$

$$\Delta_i^2 = \big(\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_{i-1}]\big)^2$$

$$Z = \sum_{i=1}^{n} \Delta_i$$

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2$$

$$= \mathbb{E}\Big[\sum_{i,j} \Delta_i \Delta_j\Big] - \Big(\sum_{i=1}^{n} \mathbb{E}\Delta_i\Big)^2$$

$$= \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2] - \Big(\sum_{i=1}^{n} \cancel{\mathbb{E}\Delta_i}\Big)^2$$

0

Tower property in the easy case:

POLL: Have you taken 403?

A = YES

B = NO

$$\int f(x_1,\dots,x_n)\, dF(x_{i+1})\cdots dF(x_n)$$
$$= \mathbb{E}[f \mid \mathcal{F}_i] = \mathbb{E}[f \mid X_1,\dots,X_n]$$

$\boxed{\text{If } \mathcal{F}_{i-1} \subset \mathcal{F}_i}$ (meaning every set in $\mathcal{F}_{i-1}$ is in $\mathcal{F}_i$)

$\mathcal{F}_{i-1} = \sigma(X_1,\dots,X_{i-1})$ $\qquad$ $\mathcal{F}_i = \sigma(X_1,\dots,X_i)$

$X_i : \Omega \to \mathbb{R}$ $\qquad$ $\sigma(X_i) = \{X_i^{-1}(B) : B \text{ is Borel}\}$

$\sigma(X_i, X_{i+1})$ = smallest σ algebra containing $\sigma(X_i)$ and $\sigma(X_{i+1})$

$$\mathbb{E}[f \mid \mathcal{F}_{i-1}] = \mathbb{E}\big[\mathbb{E}[f \mid \mathcal{F}_i] \mid \mathcal{F}_{i-1}\big]$$

$$\int f(\underbrace{x_1\dots}_{}x_n)\, dF(x_i)\cdots dF(x_n) = \int \left[\int f(\underbrace{x_1\cdots x_n}_{x_1\cdots x_i})\, dF(x_{i+1},\dots x_n)\right] dF(x_i)$$

freezing $x_1\cdots x_{i-1}$ $\qquad$ unfreeze x_i

If $g(X_1,\dots X_i)$ $\qquad \mathbb{E}[gh \mid \mathcal{F}_i] = g\,\mathbb{E}[h \mid \mathcal{F}_i]$

$$\mathbb{E}[\Delta_i] = \mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_{i-1}]\big] = \mathbb{E}\big[\mathbb{E}[Z \mid \mathcal{F}_i] - \mathbb{E}[Z \mid \mathcal{F}_{i-1}] \mid \mathcal{F}_0\big]$$

$X_1\dots X_i$ $\qquad$ $X_1\dots X_{i-1}$

$$= \mathbb{E}[Z] - \mathbb{E}[Z] = 0$$

$$\Delta_i^2 = \Big(\mathbb{E}\big[\, Z - \underbrace{\int Z\, dF(x_i)}_{E_i[Z]} \,\big|\, \mathcal{F}_i \big]\Big)^2$$

$E_i \equiv$ Expectation wrt i^{th} variable

$$\Delta_i^2 = \mathbb{E}\Big[\, Z - E_i Z \,\Big|\, \mathcal{F}_i \Big]^2$$

integration wrt i^{th} variable

$$\leq \mathbb{E}\big[(Z - E_i Z)^2 \mid \mathcal{F}_i\big]$$

Take expectation to get $\mathbb{E}[\Delta_i^2]$

$$\Rightarrow \operatorname{Var}(Z) \leq \sum_{i=1}^n \mathbb{E}\,\mathbb{E}\big[(Z - E_i Z)^2 \mid \mathcal{F}_i\big]$$

Tower property

$$= \sum_{i=1}^n \mathbb{E}\big[(Z - E_i Z)^2\big]$$

$\int \quad dF_1 \cdot dF_i \; dF_n$

$$= \sum_{i=1}^n \mathbb{E}\, E_i\,(Z - E_i Z)^2 = \sum_{i=1}^n \mathbb{E}\operatorname{Var}_i(Z)$$

depends on $X_j,\ j \neq i$

do integration over i^{th} variable first

Variance wrt just the i^{th} variable while keeping other frozen.

I have to apply conditional Jensen.

$$\mathbb{E}[Z \mid \mathcal{F}_{i-1}] = \int Z\, dF_i(x_i)\cdots dF_n(x_n)$$

$$= \int \underbrace{\int Z\, dF_{i+1}\cdots dF_n}_{\mathbb{E}[Z \mid \mathcal{F}_i]}\, dF_i$$

$$=$$

$$\Delta_i = \mathbb{E}[Z \mid \mathcal{F}_i] - E_i\big[\mathbb{E}[Z \mid \mathcal{F}_i]\big]$$

$$\int \cdot \; dF_i(x_i)$$

$\phi: \mathbb{R} \to \mathbb{R}$ convex

Z is some r.v.

$$\mathbb{E}[\phi(Z) \mid \mathcal{F}_i] \geq \phi(\mathbb{E}[Z \mid \mathcal{F}_i])$$

$$\phi(x) = x^2$$

$$\mathbb{E}\big[\mathbb{E}[\phi \mid \mathcal{F}_i]\big] = \mathbb{E}[\phi]$$

$Z - Z_i$: i^{th} variable replaced by an independent copy.

Lemma: Let X_1, X_2 be indep copies of X. Then

$$\operatorname{Var}(X) = \frac{1}{2}\mathbb{E}\big[(X_1 - X_2)^2\big]$$

Pf: (Exercise. Simply definition checking)

$\rightarrow$ expand the square, take expectation and use independence.

$$= \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\, E_{i,i'}\left[(Z - Z_i)^2\right] = \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left[(Z-Z_i)^2\right]$$

Integration over the rest of the variables.

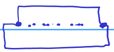
$$\mathrm{Var}_i\, Z = \frac{1}{2} \mathbb{E}_{i,i'} (Z - Z_i)^2$$

integration over X_i and the indep. copy X_i'

Proof of Kesten's Theorem

$\delta = T(0,x)$, if $E[\tau_e^2] < \infty \Rightarrow \mathbb{E}[\delta^2] < \infty$.

2nd moment of passage time exists.



$\mathbb{E}[T(0,x)^k] < \infty$

$\Leftrightarrow \mathbb{E}[\min\{\tau_1, \dots \tau_{2d}\}^k] < \infty$

Why is that?

The Efron-Stein equality extends immediately to ∞ly many variables.

Enumerate all the edges of $\mathbb{Z}^2$ as $e(1), e(2), \dots$

$T_i(0,x) \doteq$ Passage time with edge i replaced by an indep copy.

$$\mathrm{Var}(T(0,x)) \le \sum_{i=1}^{\infty} \mathbb{E}\left[(T_i(0,x) - T(0,x))^2 \mathbb{1}_{\{T_i(0,x) > T(0,x)\}}\right] \quad \text{— (\#1)}$$

symmetry

$T_i(0,x) > T(0,x)$ if

1) $i \in \underbrace{\bigcap_{\gamma} \{i \in \gamma, \gamma \text{ is a geodesic}\}}_{GEO(0,x)}$ belongs to all geodesics from 0 to x

$GEO(0,x)$ is the intersection of all geodesics from 0 to x.

2) $\tau'_{e(i)} > \tau_{e(i)}$

$\tau'_{e(i)} > \tau_{e(i)}$ $\quad \mathbb{1}_{\{e(i) \in GEO(0,x)\}}$

GEODESIC is an optimal path from 0 to x. It achieves the minimal passage time.

$$(\#1) = \mathbb{E}\left[(T_i(0,x) - T(0,x))^2 \mathbb{1}_{\{i \in GEO(0,x)\}} \mathbb{1}_{\{\tau'_{e(i)} > \tau_{e(i)}\}}\right] \quad \text{— (\#2)}$$

On this event

$$T_i(0,x) - T(0,x) \le \tau'_{e(i)} - \tau_{e(i)} \le \tau'_{e(i)}$$

So (#2) $\le \sum_{i=1}^{\infty} \mathbb{E}\left[(\tau'_{e(i)})^2 \, \mathbb{1}_{\{e(i) \in GEO(0,x)\}}\right]$

independence

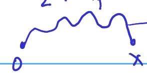
does not depend on the value of the $\tau'_{e(i)}$ variable. It only wants $e(i)$ to be in all the geodesics from 0 to x.

$$= \sum_{i=1}^{\infty} \underbrace{\mathbb{E}\left[(\tau'_{e(i)})^2\right]}_{\text{2nd moment finite}} \mathbb{E}\left[\mathbb{1}_{\{e(i) \in GEO(0,x)\}}\right]$$

$$= C\, \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{e(i) \in GEO(0,x)\}}\right]$$

$$= C\, \mathbb{E}\left[\,|GEO(0,x)|\,\right]$$

$$\mathrm{Var}(T(0,x)) \le C\, \mathbb{E}\left[\,|GEO(0,x)|\,\right] \le C\, \mathbb{E}\left[T(0,x)\right] \le C_2 \|x\|_1$$



$O(|x|_1)$

We will prove this at a later stage.

Lemma 3.3 Assume $F(0) < p_c$. $\exists$ a constant C st $\forall x \in \mathbb{Z}^d$

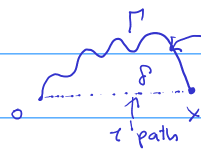
$$\mathbb{E}|GEO(0,x)| \le C\, \mathbb{E}\, T(0,x)$$

A similar result was proved by Kesten originally (Aspects 5.2)

Easy to see for bounded weights: $0 < a \le \tau_e \le b < \infty$

$$|GEO(0,x)| \le \frac{b}{a} |x|_1$$

(We have seen this in a previous lecture)



$T(\Gamma) \ge a|\Gamma|$

$T(\gamma) \le b\|x\|_1$

$T(\Gamma) \le b|x|_1$

$a|\Gamma| \le b|x|_1$

We have shown before that

$$\mathbb{E}\, T(0,x) \le C|x|_1,$$ a long time ago under

the assumption of $\mathbb{E}\left[\min(\tau_1, \dots, \tau_d)\right] < \infty$.

Lower bound:

The way to think of this is: "variance reduces if you average the function a little bit first"

$\mathrm{Var}(T(0,x)) \geq c_1 \quad \forall\, |x|,$

$\mathbb{E}[(f-\mathbb{E}f)^2] = \mathrm{Var}(f) = 0$

$\Rightarrow f = \mathbb{E}f$

Let Σ be any σ-algebra

$$\mathrm{Var}\, T(0,x) = \mathbb{E}\left[(T(0,x) - \mathbb{E}\,T(0,x))^2 \right]$$

introduce a new conditioning using the Tower property.

$$= \mathbb{E}\left[\mathbb{E}\left[(T(0,x) - (\mathbb{E}\,T(0,x))^2 \,\middle|\, \Sigma \right] \right]$$

$\mathbb{E}[\phi] = \mathbb{E}[\mathbb{E}[\phi \mid \Sigma]]$

$\mathbb{E}[\phi(X) \mid \Sigma] \geq \phi(\mathbb{E}[X \mid \Sigma])$

$$\geq \mathbb{E}\left[\left[\mathbb{E}[T(0,x) \mid \Sigma] - \mathbb{E}\,T(0,x) \right]^2 \right]$$

Jensen

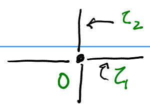
$\mathbb{E}\left[\mathbb{E}\left[T(0,x) - \mathbb{E}[T(0,x)] \,\middle|\, \Sigma \right]^2 \right]$

constant

$$= \mathrm{Var}\left(\mathbb{E}[T(0,x) \mid \Sigma] \right)$$

$\mathbb{E}\left[(\underbrace{\mathbb{E}[T(0,x) \mid \Sigma]}_{Y} - \underbrace{\mathbb{E}[T(0,x)]}_{\mathbb{E}Y})^2 \right]$

(#4)



Choose this conditioning $\to$ Choosing some variables to freeze

POLL

I was able to follow this computation

YES OR NO

let $\Sigma = \sigma(\tau_1, \ldots, \tau_{2d})$

$\Sigma' = \sigma(\tau_1', \ldots, \tau_{2d}')$ be an indep copy.

Then $\mathbb{E}[T(0,x) \mid \Sigma] = f(t_1, \ldots, t_{2d})$

is a function of the random vars $\tau_1, \tau_2, \ldots \tau_{2d}$

and $\mathbb{E}[T'(0,x) \mid \Sigma'] = f(t_1', \ldots, t_{2d}')$

where $T'(0,x)$ is the passage time where $t_1 \cdots t_{2d}$ are replaced with $t_1' \cdots t_{2d}'$ (or all of the weights)

$$\mathrm{Var}(X) = \frac{1}{2}\,\mathbb{E}\big[(\underbrace{X_1 - X_2}_{\text{indep copies}})^2\big]$$

can be replaced. It doesn't make a huge difference.)

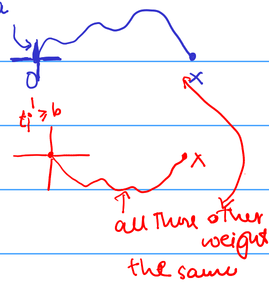
Thus using our previous lemma

$$\mathrm{Var}\big(E[T(0,x)\mid\Sigma]\big)$$

$\tau_1' \ldots \tau_{2d}'$ are all small.

$$= \frac{1}{2}\,\mathbb{E}\Big[\big(E[T(0,x)\mid\Sigma] - E[T'(0,x)\mid\Sigma']\big)^2\Big] \quad - \#3$$

$\tau_1 \ldots \tau_{2d}$ are all large



Fix $a < b$ st $P(\tau_e \le a) > 0$ and

$P(\tau_e \ge b) > 0$

On the event A, $t_i \le a$ and $t_i' \ge b$ $\forall i = 1, \ldots, 2d$

$$\mathbb{E}[T'(0,x)\mid\Sigma'] - \mathbb{E}[T(0,x)\mid\Sigma] \ge (b-a)$$

Σ': choose some set of variables to condition over.

Thus

$$(\#3) \ge \frac{1}{2}\,\mathbb{P}(A)\,(b-a)^2 = C$$

integrating over a smaller set.

$$\ge \frac{1}{2}\,\mathbb{E}\Big[\big(E[T(0,x)\mid\Sigma] - E[T'(0,x)\mid\Sigma']\big)^2 \mathbb{1}_A^2\Big]$$

$$\ge \frac{1}{2}(b-a)^2\,\mathbb{E}[\mathbb{1}_A]$$

$\mathrm{Var}(T(0,x)) \ge C$ (independent of $|x|_1$)

$$\mathbb{P}(A) = \mathbb{P}(t_1 \le a, \ldots t_{2d} \le a,\ t_1' \ge b, \ldots t_{2d}' \ge b)$$

$$\prod_{i=1}^{2d} \mathbb{P}(t_i \le a)\,\mathbb{P}(t_i' \ge b) > 0$$

Obvious question: Can you improve this bound somehow?

Newman and Piza provided the ONLY improvement since then ($\log n$), in the 1990s. This was only in $d=2$.

Since then there has been NO PROGRESS.

Question: Can you improve the conditioning to improve the lower bound to n^{ϵ} for any $\epsilon > 0$? $(\log n)^2$

We've proved

$$c \leq \mathrm{Var}(T(0,x)) \leq C|x|_1 \quad \text{using Efron-Stein / discrete Poincare.}$$

$$\mathrm{Var}(T(0,x)) \approx |x|^{2/3}$$

Random matrices
Zeta fn.

(2003) Benjamini - Kalai - Schramm.

(1994) Michel Talagrand : Ann. of Prob.

(1997) " : Isoperimetric ineq. Publication of IHES.

1993 Kesten

1988 (Kahn-Kalai-Linial)
Theoretical CS
"Influence ineq. on Boolean fns"
$f : \{0,1\}^n \to \mathbb{R}$
(1992) (Bourgain - Katznelson - Kahn . .)

"discrete Fourier analysis"

Used this inequality to

$$\mathrm{Var}(T(0,x)) \leq \frac{|x|_1}{\log |x|_1} \qquad \tau_e \in \{a,b\}$$

POLL : Have you heard of this Guy.

YES OR NO

Do you guys want to hear a few words about the history?

YES OR NO