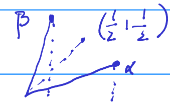
lec 08

Now we need a so-called coupling argument.
The graphical construction allows us to couple two initial conditions: fix the randomness (the red sites) and then run the two processes with different initial data.

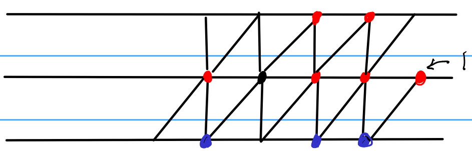


The coupling here refers to the fact that they are run on the same state space.

$$\frac{r_n(p+\delta)}{n} - \frac{r_n(p)}{n} \geq \delta$$ then we have that $\alpha - \beta \geq 2\delta$

Recall we want to compare $\xi_n^{p+\delta}$ and ξ_n^{p}; in particular we want to compare $r_n^{p+\delta}$ and r_n^p.

$$r_n^{B\cup\{0\}}(p) - r_n^{B}(p)$$



Space: $\{U_z\}_{\mathbb{Z}\times\mathbb{Z}^+}$, $\mathbb{P}^{\otimes \mathbb{Z}\times\mathbb{Z}^+}$

$\mathbb{P}(U_z \leq p) = p$

If you want to build both processes on the same space, let's throw $\{U_z\}_{z\in\mathbb{Z}\times\mathbb{Z}^+}$ $U_z \sim \mathrm{Unif}[0,1]$

iid rvs. Then a particle is red for $\xi_n^{p+\delta}$ if

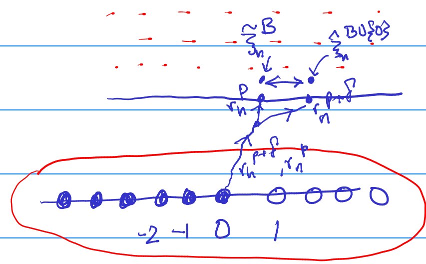
$U_z \le p+\delta$ and its red for ξ_n^{p} if

$U_z \le p$

Thus we have "coupled" $\xi_n^{p+\delta}$ and ξ_n^{p}

Fix initial data to be

$$1_{(-\infty,0]}(x) = \xi_0(x) \qquad \text{for } x \in \mathbb{Z}.$$



On this space, I claim that

$$\xi_n^{p+\delta}(x) \ge \xi_n^{p}(x) \qquad \forall n,x$$

(A red path in env. p is a red path in env. $p+\delta$)

So $r_n^{p+\delta} \ge r_n^{p}$.] Rightmost particle in $\xi_n^{p+\delta}$ is to the right of ξ_n^{p}

stopping time

let $\tau = \inf\{m : r_n^{p+\delta} > r_n^{p}\}$

(First time at which right most particles separate)

So in this coupled system we can think of each site in $\mathbb{Z}\times\mathbb{Z}^+$ as RED_1 (for just $\xi_n^{p+\delta}$) or RED_2 (for both processes)

$\mathbb{E}\left[r_n^{B\cup\{0\}}(p) - r_n^{B}(p) \right]$

Define a 3rd process $\hat{\xi}_n$ which follows RED_1 sites until time τ (a stopping time) and then RED_2 there after.

a combination of $\xi_n^{p+\delta}$ and ξ_n^{p}

So $\xi_n^{p+\delta}(x) \geq \hat{\xi}_n(x) \geq \xi_n^{p}(x)$

The idea is that ONCE $r_n^{p+\delta}$ and r_n^{p} separate by 1, we can apply the earlier bound in (#3).

rightmost particle

Let $\hat{r}_n = \sup\{y : \hat{\xi}_n(y) = 1\}$

$\hat{r}_n - r_n^{p} = (r_n^{p+\delta} - r_n^{p})\, 1_{\{\tau \geq n\}}$

they have not already split.

$E[r_n^{p+\delta} - r_n^{p}] \geq E[\hat{r}_n - r_n^{p} ; n \geq \tau]$

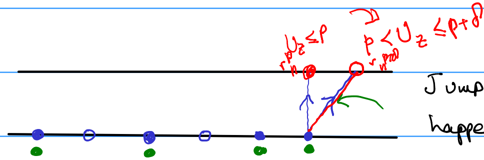
worth thinking about for a second.

Stopping time: Durrett's text book.

independence.

But $E[\hat{r}_n - r_n^{p} ; n \geq \tau] \geq 1 \cdot P(n \geq \tau)$ from (#3). (We can think of the new process starting anew at time τ.)

$P(\tau > n) = P($ right most particle in $\xi^{p+\delta}$ never jumps ahead of the right most particle in $\xi^{p})$



Jump ahead happens if

$p < U_z \le p+\delta$

which has prob δ.

$1 - \mathbb{P}(p < U_z \le p+\delta)$

$= 1-\delta$

$P(\tau > n) = (1-\delta)^n$

$\Rightarrow P(\tau \le n) = 1 - (1-\delta)^n$

$$\mathbb{E}\Big[\underbrace{r_n^{p+\delta}}_{n} - \underbrace{r_n^{p}}_{n}\Big] \geq \Big(1 - \overbrace{(1-\delta)^n}^{e^{-\delta n}}\Big) \geq \delta$$

$\frac{\delta}{n}$

Next trick.

$$\mathbb{E}\left[r_n^{p+\delta} - r_n^{p+\delta-\frac{\delta}{k}}\right] \geq 1 - (1-\delta/k)^n$$

$$\mathbb{E}\left[r_n^{p+\delta-\frac{\delta}{k}} - r_n^{p+\delta-\frac{2\delta}{k}}\right] \geq 1 - (1-\delta/k)^n$$

$\mathbb{E}$ ⋮

$$= \mathbb{E}\left[r_n^{p+\delta} - r_n^{p}\right] \geq k\left(1 - \left(1-\frac{\delta}{k}\right)^n\right) \quad \text{— (\#4)}$$

Then let $k \to \infty$

$$\approx k\left(1 - e^{n\log\left(1-\frac{\delta}{k}\right)}\right)$$

$$\approx k\left(1 - e^{n\left(-\frac{\delta}{k} - \frac{\delta^2}{2k^2}\right)}\right) \approx k\left(1 - 1 + \frac{n\delta}{k} + \frac{n\delta^2}{2k^2}\right)$$

$\longrightarrow n\delta$

Then (#4) becomes

$$\frac{1}{n}\mathbb{E}\left[r_n^{p+\delta} - r_n^{p}\right] \geq \delta$$

Take a limit

$\updownarrow$ requires $p > p_{cr}$ $\longleftarrow$ previous theorem on directed percolation due to Durrett.

Get $\alpha(p+\delta) - \alpha(p) \geq \delta$

By symmetry $\overbrace{\beta}^{\text{leftmost particle}}(p+\delta) - \beta(p) \leq -\delta$ for $p > p_{cr}$

l_n goes left. Thus

$$\alpha(p+\delta) - \beta(p+\delta) - (\alpha(p) - \beta(p)) \geq 2\delta$$

$\Longrightarrow$ $\overbrace{\alpha(p) - \beta(p)}^{\text{Then there is a flat interval on the Crit shape}} \geq 2(p - p_{cr})$

and let $p \downarrow p_{cr}$ so that $\alpha(p) - \beta(p) \longrightarrow 0$ (by definition)

This gives us the existence of the cone.

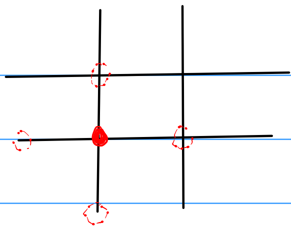
The paper also talks about

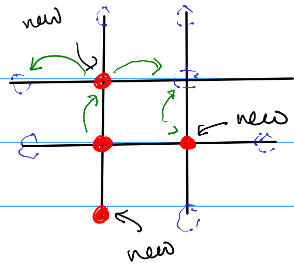
1) Branching Random Walk
2) Lower bounds for p_{cr} ($\geq \frac{1}{2}$, ≥ 0.618 etc.)

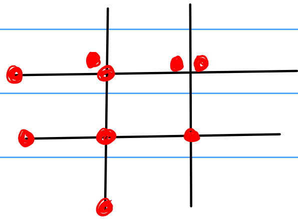
describe the extents of this cloud of particles in the BRW.

This shows how you can bound Richardson's growth process ABOVE using the BRW.

BRW (In 2d)







You see how there is no interaction? Thats an advantage.

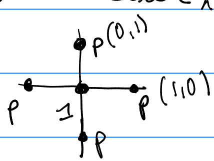
It also grows faster since each particle can invade the next region.

Let $N_n(x)$ = # particles at x at time n.

set of pts with particles in Branching process.

$$\{x : N_n(x) \geq 1\} \supseteq \underbrace{\{x : \eta(x) = 1\}}_{\text{particles in Richardson's process}}$$

m_p is a measure, on $\mathbb{Z}^2$



MGF or Laplace transform that's 2D.

$$k(\theta) = \log \int e^{-\langle\theta, x\rangle} \, d\mu_p(x) \qquad \theta \in \mathbb{R}^2$$

$$= \log \left[p e^{-\theta\cdot(1,0)} + p e^{-\theta\cdot(0,1)} + p e^{-\theta\cdot(-1,0)} + p e^{-\theta\cdot(0,-1)} + e^{-\theta\cdot(0,0)} \right]$$

concave

Legendre transform. (convex dual)

$$k^*(y) = \inf_{\theta} \left\{ k(\theta) + \theta\cdot y \; : \; \theta \in \mathbb{R}^2 \right\}$$

Theorem (Biggins 1978):

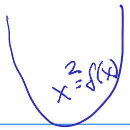
Let H_n = convex hull of $\{x : N_n(x) > 0\}$

$D = \{ y : \underbrace{k^*(y)}_{\text{EXPLICIT function.}} > 0 \}$

$$\underline{\lim} \frac{H_n}{n} = \overline{\lim} \frac{H_n}{n} = D$$

(The limits are in the same sense as the limit shape theorem)

Cool theorem because the limit shape is explicit.



$$L(x) = \sup_p \left(px - f(p)\right)$$

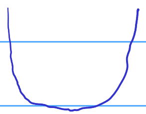
$$\frac{\partial}{\partial p}\left(px - p^2\right) = x - 2p, \qquad p = \frac{x}{2}$$

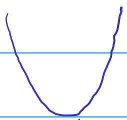
$$L(x) = x \cdot \frac{x}{2} - \frac{x^2}{4} = \frac{x^2}{4}$$

$f(x) = x^4$ $\qquad L(x) = \sup_p \left(px - p^4\right)$ $\qquad 4p^3 = x$

$$p = \left(\frac{x}{4}\right)^{1/3}$$

$$= \frac{x^{4/3}}{4^{1/3}} - \frac{x^{4/3}}{4^{4/3}} = C x^{4/3}$$

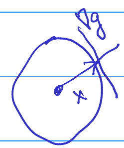




g'

$$f(g'(x)) = g' \cdot x$$

$$g(x) + f(p) = p \cdot x$$



Mp, $D \subset \mathcal{B}_1 \leftarrow$ e' ball

Theorem: If $p < \frac{1}{2}$ then

$\mathcal{B} \cap \partial\mathcal{B}_1 = \phi$

Recall that if $p > p_{cr}$ then $\mathcal{B} \cap \partial\mathcal{B}_1$ is an interval. So this shows that

$$p_{cr} \geq \frac{1}{2}$$

I will not pursue the proofs of these theorems since they're mostly computational, but it comes from the fact that $k^*(y)$ is

EXPLICIT.

☆ Student reading: How does Biggins prove his theorem.