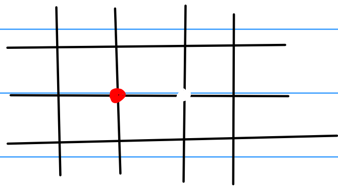
Durrett Liggett 1981

Richardson's Model.

1) All points in $\mathbb{Z}^2$ are white at time 0, except origin which is red



2) At any subsequent time step, each white site in $\mathbb{Z}^2$ checks if any of its neighbors is red. If it is, it flips a coin; if it turns up heads, it turns red. Otherwise it remains white.

3) If none of its neighbors are red, it remains white

4) If its red, it remains red.

State : $\eta_n(x) = \begin{cases} 1 & x \text{ is red at time } n \\ 0 & \text{white at time } n \end{cases}$

$\eta_n : \mathbb{Z}^2 \longrightarrow \{0,1\}$ $\qquad \{\eta_n\}$ is a Markov Process

η_n depends only on η_{n-1} and the iid outcomes ~~at~~ white sites that neighbor red sites.

We can extend ρ to $\mathbb{R}^2$ like we extended $T(x,y)$ to $\mathbb{R}^d \times \mathbb{R}^d$

Let $A_n = \{x : \rho_n(x) = 1\}$

Theorem (Richardson) 1973 : $\exists$ $\mathcal{B}(\rho), \mu(x)$, st for any $\epsilon > 0$,
$\mathcal{B}(\rho) = \{x \mid \mu(x) \leq 1\}$

$$\lim_{n\to\infty} \mathbb{P}\left(\{x : \mu(x) \leq 1-\varepsilon) \subseteq \frac{A_n}{n} \subseteq \{x : \mu(x) \leq 1+\varepsilon\} \right) = 1$$

What is $\mu(x)$? It's a norm. Let

$$t_0(x) = \inf\{n \mid \rho_n(x) = 1\}$$

Then $$\mu(x) = \inf_n \frac{\mathbb{E}\, t_0(nx)}{n}$$ Subadditive ergodic theorem. we haven't proved this as yet.

<u>Remark</u>: Richardson's Theorem was the "convergence in probability" version of Durrett and Liggett
Richardson's model to Geometric FPP (vertex weights)

They exploit the relationship between

easier to understand
1) Contact Processes
lower bound
2) Directed Percolation
upper bound
3) Branching Random Walk.
Donggeun will tell us more soon.

* Show pictures from Richardson's Paper.

Site percolation: $\{\omega_x\}_{x \in \mathbb{Z}^2}$ iid, $\gamma = \{x_0, \dots, x_n\}$

$$W(\gamma) = \sum_{i=1}^{n} \omega_{x_i} \qquad T(x,y) = \inf_{\gamma : x \to y} W(\gamma)$$

Is the relationship between Richardson's model and Geometric FPP obvious?

Proposition If $\{\omega_x\}$ are iid (shifted) geom. taking values $\{1, 2, \dots\}$ such that

$$\mathbb{P}(\omega_x = k) = (1-p)^{k-1} p \qquad k = 1, 2, \dots$$

Then $T(0,x)$ has the same distribution as $t_0(x)$ (time at which a certain site turns red)

$$t_0(x) = \inf \{ n \mid \underbrace{\xi_n(x) = 1}_{x \text{ turns red for the 1st time}} \}$$

Pf: Let $t_b(x) = \min_{y \sim x} \{ t_0(y) \}$ (The 1st time at which one of the neighbors of x becomes red in Richardson's model)

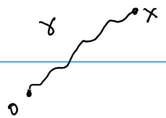
(1st time at which a neighbor turns red)

Then $t_0(x) - t_b(x) =$ 1st time at which independent coin flips turns up heads $=$ Geom (p)

Moreover $\{t_0(x) - t_b(x)\}_{x \in \mathbb{Z}^2}$ are iid geometric

$\omega_x = \{t_0(x) - t_b(x)\}_{x \in \mathbb{Z}^2}$ iid Geometrics.

Take



smallest time at one of the neighbours of x_i

$t_b(x_i) \le t_0(x_{i-1})$

weight of any path

$$\omega(\gamma) = \sum_{i=1}^{n} \omega_{x_i} = \sum_{i=1}^{n} t_0(x_i) - t_b(x_i)$$

$$\geq \sum_{i=1}^{n} t_0(x_i) - t_0(x_{i-1}) = t_0(x) - \cancel{t_0(0)}^{\,0}$$

Subtracting something larger

$\omega(\gamma) \geq t_0(x)$

$T(0,x) \geq t_0(x)$

But you could also define γ recursively by choosing

$$x_{i-1} = \underset{y \sim x_i}{\operatorname{argmin}} \{t_0(y)\}$$

Choose x_{i-1} such that $t_b(x_i) = t_0(x_{i-1})$

For this this particular path $\omega(\gamma) = t_0(x)$

$T(0,x) =$

(Cox, continuity)

<u>Theorem</u>: Let $F_n \to F$ and $1 - F_n(x) \le 1 - U(x)$ st U has finite mean. Then

$$\mu_{F_n}(x) \to \mu_F(x)$$

why do we need this? (Uniform integ. type condition)

$$\int_0^\infty (1 - U(x))\,dx = \int_0^\infty x\,U'(x)\,dx = \text{mean}.$$

The distribution F_n converges to F (weak-$*$) then time constants converge.

$F_n(x) \to F(x)$ at all continuity pts of F_0

<u>Exercise</u>: Let $\mu_p(x)$ be the time constant

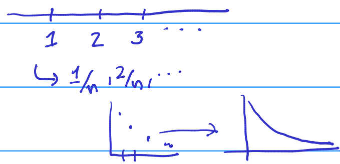
when $w_x \sim \text{Geom}(p)$ iid.

Then $\mu_p(x)$ is a continuous fn of p (for any fixed x)

Cox continuity.

rescales to an exponential

Rescaling the geometric: let $X \sim \text{geom}(p)$



$$\mathbb{P}(X = k) = (1-p)^{k-1} p \qquad k = 1, 2, \dots$$

$$\mathbb{P}(X > k) = \sum_{s=k+1}^{\infty} p(1-p)^{s-1} = p(1-p)^k \sum_{s=0}^{\infty} (1-p)^s = (1-p)^k$$

$$\mathbb{P}(pX > pk) = (1-p)^{k} \qquad \frac{t}{p} \qquad e^{-t} \qquad k = p, 2p, 3p, \dots$$

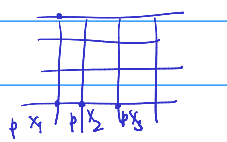
So for any fixed t, write it as $t = pk + \epsilon$, $0 < \epsilon < p$

$$\lim_{p \to 0} \left(1 - \frac{x}{n}\right)^n \to e^{-x} \quad \text{So}$$

$$P(pX > t) = (1-p)^{\frac{t-\epsilon}{p}} \longrightarrow e^{-t}$$

pX is converging in distribution to an exponential as $p \downarrow 0$

$p\mu_p(x)$



$\mu_{pX}(t) = p\mu_X(t)$

Let $H(x) = (1 - e^{-x})^+$ be the cdf of an exponential

and μ_H be the time constant. Then,

time const for geom.

time constant for exponential wts.

<u>Theorem</u>: $\lim_{p \downarrow 0} p\mu_p(x) = \mu_H(x)$

Simulations, Richardson observed as $p \to 0$, the curve looks "more and more" like a circle or ball

Richardson's conjecture: $\mu_H(x) = \sqrt{x^2 + y^2}$

Open.

$p = 1$ when all weights are 1

Note that $M_1(x) = |x|_1$ the ℓ^1 norm.

Most people think this conjecture is false.

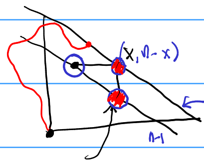
bunch of papers in the Annals of Prob. | Smirnov, Neiner

Durrett (Duke), Textbook 406, 1980s: Contact process, directed percolation & so on
↳ very well written.

Contact Process

Richardson's process

$\xi_n(x) = R_n(x, n-x)$



focus on the problem in the positive quadrant.

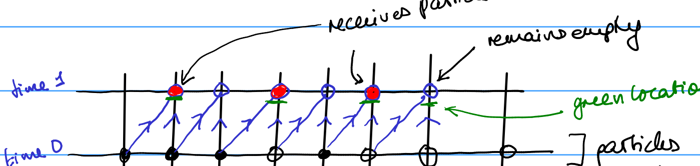
$\xi_n(x) = 1$ if

$\xi_{n-1}(x) = 1$ or $\xi_{n-1}(x-1) = 1$ and $\omega_{(x,n-x)} = 1$ (with prob p)

Otherwise $\xi_n(x) = 0$.

↓ directly related to the

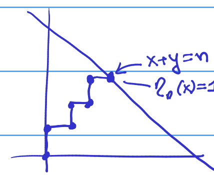
Contact Process in 1D



green locations are where my geometrics turn up with $\omega_x = 1$

Initial condition: $\xi_0(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$

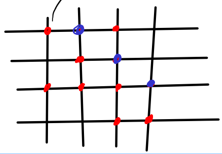
Richardson's process dominates this contact process.



focuses only on upright percolating paths.

$\{x \in \mathbb{Z}^2 \mid R_n(x) = 1\} \supseteq \{(y, n-y) : \xi_n(y) = 1\}$

Obviously, because



let $\Omega_\infty = \{\omega : \xi_n \not\equiv 0 \quad \forall n\}$ (not identically 0)

= " $\exists$ an ∞ up-right path from the origin to ∞ "

Oriented percolation.

Theorem (Harris 1978 AoP)

$\exists$ a $0 < p_0 < 1$ st for $p > p_0$ $P(\Omega_\infty) > 0$

If $P(\Omega_\infty) > 0$ Then

$A_n = \{x : \eta_n(x) = 1\}$ always contains a point (x,y) st $x+y = n$

$\mathbb{P}\left((1-\varepsilon)\mathcal{B} \subseteq \frac{A_n}{n} \subseteq (1+\varepsilon)\mathcal{B}\right) \longrightarrow 1$

Hence $\mathcal{B}$ must intersect the e^1 ball, for if not

$$\mathbb{P}\left(\frac{A_n}{n} \subset (1+\varepsilon)\mathcal{B}\right) \not\longrightarrow 1$$

In fact, by symmetry if $(x_0, 1-x_0) \in \mathcal{B}$, $(1-x_0, x_0) \in \mathcal{B}$

and by convexity

$$\frac{1}{2}(x_0, 1-x_0) + \frac{1}{2}(1-x_0, x_0) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathcal{B}$$

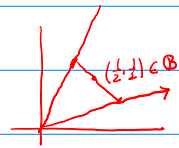
Let $\quad p_{cr} = \inf\{p \mid P(\Omega_\infty) > 0\}$

Theorem (Durrett): If $p > p_{cr}$ then $(\frac{1}{2}, \frac{1}{2}) \in \mathcal{B}$.

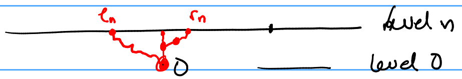
$$\mu(\tfrac{1}{2}, \tfrac{1}{2}) = 1$$

Remark: This is the first (and only) case in which we can exactly identify $\mu(x_0)$ for some x_0 in FPP.

Lets analyze this contact process a little more. (Seviah)



Define



p_{cr}

$p > p_{cr}$

size of the line segment $> c(p - p_{cr})$

Is there a graph where you can solve the particles system.

$r_n = \sup\{y : \xi_n(y) = 1\}$

$l_n = \inf\{y : \xi_n(y) = 1\}$

$\Omega_n = \{\omega : \xi_n \neq 0\}$.

the event on which there is at least one particle at level n.

The contact process has not died out.

Theorem (Durrett, On the growth of 1D contact process) 1980s

$$\left.\begin{aligned} \frac{r_n}{n} \mathbb{1}_{\Omega_n} &\longrightarrow \alpha \mathbb{1}_{\Omega_\infty} \\ \frac{l_n}{n} \mathbb{1}_{\Omega_n} &\longrightarrow \beta \mathbb{1}_{\Omega_\infty} \end{aligned}\right] \text{ a.s and in } L^1$$

r_n grows linearly with rate α

l_n " " β

Remark: I dont know why this Theorem is true, since I haven't read the paper.

I think I have a guess

This would be fun to read.

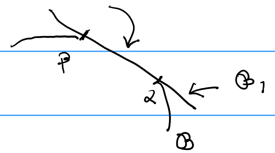
Theorem If $p > p_{cr}$ ($0 < p_{cr} < 1$) then $\alpha - \beta \geq 2[p - p_{cr}]$

Gromov - Hausdorff

Curvature of the limit shape

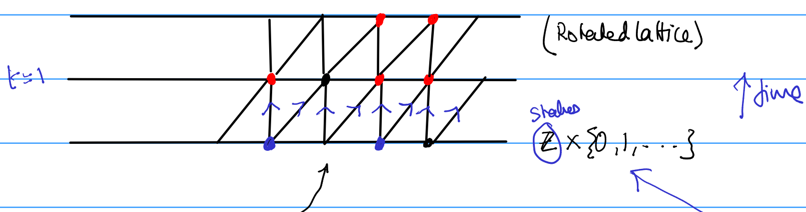
Corollary: $\partial B \cap \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 1\}$ is an interval of length at least $2[p - p_{cr}]$

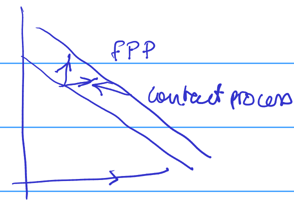
$Var(T(0,x)) \to L^2$



Proof ideas: (Contact process)

Graphical representation. (due to Harris)





- Initial configuration ξ_0
- Open sites determined by flipping iid Bernoullis

A classical coupling.

Has coupled any two $\{\xi_n^A\}$ and $\{\xi_n^B\}$

Blue dots can travel along paths of red vertices.

Then (more precisely) $\xi_n(x) = 1$ if $\exists$ a

Coupling (den Hollander, notes on Coupling)

(X, μ) (Y, ν) $(X \times Y, \rho)$ such that the marginal of $\rho_X = \mu$ (Marginal distributions)

$\rho_X \rightarrow$ pushforward under the projection map on 1st coordinate and $\rho_Y = \nu$.

$\Omega = \mathbb{R}^{\mathbb{Z}}$ $\{X_n\}$ of random variables or if $\omega \in \Omega$

$X_n = \omega(n)$. $(\Omega, \mathbb{P})$

2 stochastic processes $(\Omega_1, \mathbb{P}_1)$ $(\Omega_2, \mathbb{P}_2)$

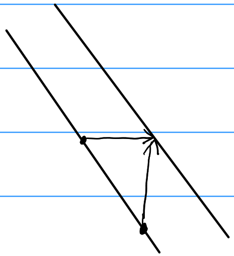
A coupling "builds them on the same space" $(\Omega_1 \times \Omega_2, \Pi)$

Boring coupling : $\mathbb{P}_1 \otimes \mathbb{P}_2$. Generally useless for comparing two stochastic processes.

Set of initial particles.

path of red vertices from $y \in A$ to x at level n

n.



It's the same as the rotated contact process we first defined.

Additivity Property: $\xi_n^{A\cup B}(x) = \underbrace{\xi_n^{A}(x) \vee \xi_n^{B}(x)}_{\text{this makes sense right?}}$

where $a \vee b = \max(a,b)$

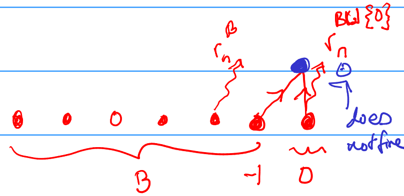
Then if $B \subset (-\infty,-1]$ (initial set of particles)

$\mathbb{E}\left[r_n^{B\cup\{0\}} - r_n^{B} \right] \geq 1.$ — (#3)

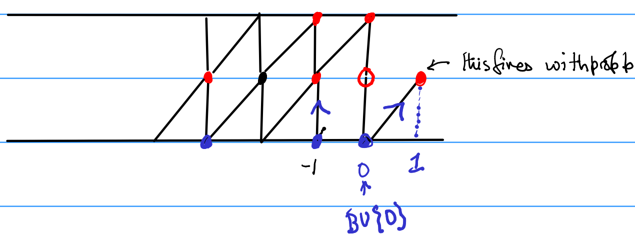
One expectation operator

Key to proving Durrett's theorem $\frac{r_n}{n} \to \beta$ $\frac{\ell_n}{n} \to \alpha$

rightmost leftmost.



This is clear too



Outcomes at time 1	B	$\{0\}$		$r_1^{B\cup\{0\}} - r_1^{B}$	Probability of staying at least one ahead after one step.
	-1	0	1		
	o	o	●	≥ 3	$p(1-p)^2 \dots$
		o	●	≥ 2	$p(1-p)$
		●	●	≥ 1	p^2
		●	o	≥ 0	$p(1-p)$

expectation $\frac{1}{p} - p$

" p^2

$$\mathbb{E}\left[r_1^{B\cup\{0\}} - r_1^{B}\right] \geq 1\, p^2 + 2p(1-p) + 3p(1-p)^2 + \dots$$

$$= p^2 + \frac{1}{p} - p$$

$X \sim \text{Geom}(p) \in \{1, 2, \dots\}$

$\mathbb{P}(X=k) = p(1-p)^{k-1}$

$\mathbb{E}[X] = p + 2p(1-p) + \dots$

Then I plotted this guy and it seems ≥ 1 so it must be true.