Lec 06 : A quick digression on passage times when w_e is allowed to be $+\infty$ or if

$$E\left[\min_{i=1,\dots,2d} \tau_i\right] = +\infty.$$

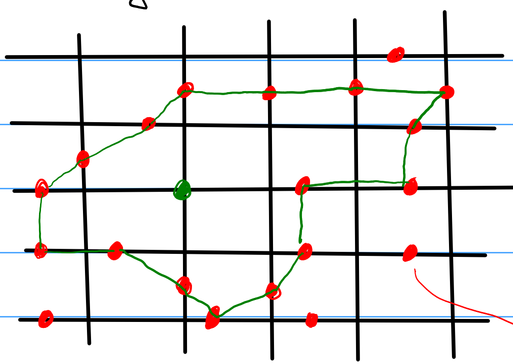
$$\overline{\lim}\ \frac{T(0,nx)}{n} \to +\infty$$

(Borel Cantelli)

$$\overline{\lim}\ \frac{1}{n}T(0,nx) < +\infty$$

$\bullet = \tau_e \leq M$

Kesten's lovely idea



Case

1) $P(\tau_e < \infty) = 1$ BUT $E\left[\min_{i=1,\dots,d} \tau_i\right] = +\infty$

2) $P(\tau_e = +\infty) > 0$ But $\exists M$ $P(\tau_e \leq M) > p_c$

By Burton & Keane there is a unique infinite open cluster.

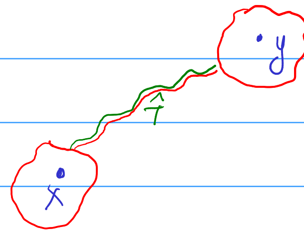
M is so large that $\mathbb{P}(\tau_e \leq M) \approx 1 \Rightarrow$ there is a very large percolation cluster of wts where $\tau_e \leq M$.

If τ_e is finite this M can always be found. If τ_e can be ∞, then we must assume this condition.

By a theorem of Burton and Keane (see list of papers) there are 2 kinds of

edges

- $P(w_e \le M)$ (open)
- $P(w_e > M)$ (closed)



$\hat{T}(x,y) \le M|x-y|_1$ (more or less with a lot missing geom details)

ONLY ONE of these can percolate (be part of an ∞ cluster); in other words there is ONLY ONE ∞ cluster

$\Rightarrow$ ∞ cluster is CONNECTED

Conclusion: Every vertex is surrounded by a shell of good edges (called S_x)

So can define $\hat{T}(u,v)$ as the

smallest passage between S_u and S_v
(there is a path taking weights $\leq M$ on the percolation cluster)

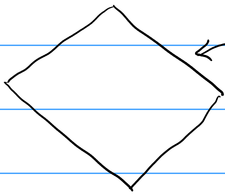
$\hat{T}(u,v)$ Can prove Kingman's theorem for the modified passage times.

Similarly FPP can be considered on the when weights are allowed to ∞
$0 < P(w_e = +\infty) < p_c$ (no percolation)

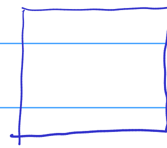
$w_e \in \{1, +\infty\}$
ideas from percolation to say more precise things about the time constant.

What types of limit shapes can you get?

We saw: $w_e = 1$ identically $\Rightarrow$



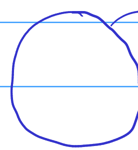
$\{x : \mu(x) = |x|_1 = 1\}$



NOT the limit shape for any class of weights.

<u>Questions</u>: Is the d-dimensional cube (ℓ^∞ ball) a limit shape for any iid weights?

Is the Euclidean ℓ^2 ball a limit shape?



Euclidean ball a limit shape for any iid weights?

Kesten says for d large enough, for a class of weights with density $\rightarrow$ (8.4, Aspects) the d dimensional sphere is NOT a limit shape.

$\mathcal{B}_r = \{x \in \mathbb{R}^d \mid |x|_2 \leq r\}$ Then $\mathcal{B}_r$ is not the limit shape $\mathcal{B}$

$(\mathcal{B} = \{x \mid \mu(x) \leq 1\})$ for any r.

D. Dhar's result about exponential percolation in large dimensions

→ Sandpiles.

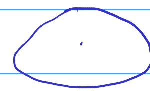
The stationary case:

When $(\Omega, \{T^z\}_{z\in\mathbb{Z}^d}, \mathbb{P})$ is merely stationary and ergodic, (not iid) then any convex shape C, symmetric about the origin can be achieved.

contain the origin (must have symmetries of $\mathbb{Z}^d$)

Haagstrom, Meester.

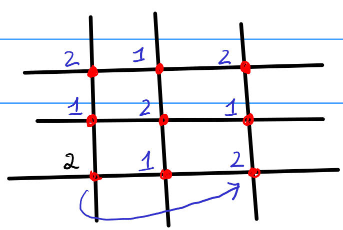
Lovely result.



(Theorem is due to Haagstrom and Meester, worth reading)

However many boring systems are captured under the stationary-ergodic umbrella.

let $\omega_1(z) = \begin{cases} 1 & |z|_1 \text{ is odd} \\ 2 & |z|_1 \text{ is even} \end{cases}$ $\qquad \omega_2(z) = \begin{cases} 1 & |z| \text{ is even} \\ 2 & |z| \text{ is odd.} \end{cases}$



$$\Omega = \{\omega_1, \omega_2\}$$

$$T^{e_k}\omega_1 = \omega_2 \qquad T^{e_k}\omega_2 = \omega_1 \qquad (k=1,2)$$

$T^{2e_1}\omega_1 = \omega_1$ $\{\omega_1\}$ is inv for T^{2e_1} but it's meas is $\frac{1}{2}$

$$\text{let } \Omega = \{\omega_1(z), \underset{\substack{\| \\ \omega_2(z)}}{\omega_1(z+e_1)}\}$$

$$\mathbb{P}(\omega_1(z)) = \mathbb{P}(\omega_2(z)) = \frac{1}{2} \Rightarrow \mathbb{P}(T^z A) = \mathbb{P}(A)$$ (Stationarity of the measure)

To define $T^z : \Omega \longrightarrow \Omega$ its enough to define the generators T^{e_1} and T^{e_2}.

These are obvious.

1) Stationarity: $\mathbb{P}(\omega_1) = \mathbb{P}(T^z \omega_1)$

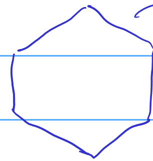
obvious.

2) Ergodicity: A is called INVARIANT (Ergodicity) if $T^z A = A \quad \forall z \in \mathbb{Z}^2$

Neither $\{\omega_1\}$ nor $\{\omega_2\}$ are invariant.

So only invariant sets are ϕ and Ω, and these are trivial. So we have ergodicity.

Periodic systems are quite boring.



Limit shape is a polygon.

Question: Are periodic limit shapes always polygons?

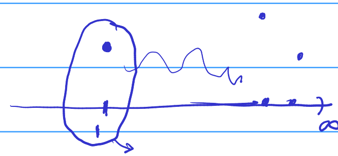
Only 1 ~~config~~ config is very trivial

2 configs is slightly less trivial.

So lets focus on the truly random iid case.

Limit shapes with a flat edge

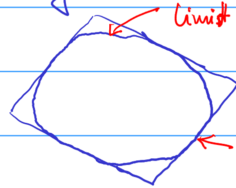
The Durrett-Liggett theorem.



Focusses on special edge wts:

$$\left.\begin{array}{l} \mathrm{supp}(\nu) \subseteq [1,\infty) \\ \nu(\{1\}) = p \geq \vec{p}_c \end{array}\right] \text{Class of measures called } M_p \text{ (percolating measures)}$$

Will define $\vec{p}_c$ shortly



$\vec{p}_c$ is the critical probability for ORIENTED PERCOLATION

→ Contact process (particle system)

→ Branching process

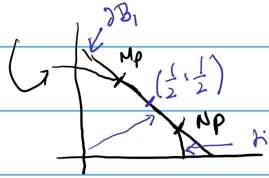
] 202 → Williams' Prob. with Martingales (5 pages)

Let α_p be the "speed of oriented percolation" (will elaborate) $\quad p = \quad \nu(\{1\})$

Then $B_1 = \{x \in \mathbb{R}^2 : |x|_1 \le 1\}$ $\quad$ (ℓ^1 ball)

$M_p = \left(\frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}\right)$

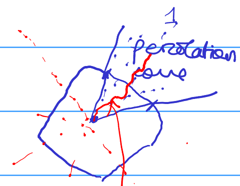
∂B



$N_p = \left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}\right)$

times are a little longer / slower percolation

$\{x \mid \mu(x) \le 1\}$ $\quad$ $\{x \mid |x|_1 \le 1\}$



Thm: (Durrett, Liggett + Marchand) $(d=2)$

1) $B \subset B_1$ $\quad$ (not as fast as fastest possible)

2) $p < \vec{p}_c$, $\quad B \subset \text{int } B_1$ $\quad$ $\mu(x) > |x|_1$

red path percolates

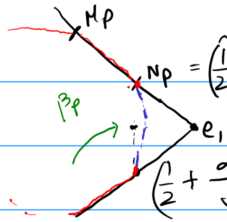
$\Rightarrow \mu_p(x) = |x|_1$

3) $p > \vec{p}_c$, $\quad B \cap \partial B_1 = [M_p, N_p]$ in positive quadrant

4) $p = \vec{p}_c$, $\quad B \cap \partial B_1 = (\frac{1}{2}, \frac{1}{2})$

was this in Durrett / Liggett?

Marchand:



$N_p = \left(\overbrace{\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}}^{\beta_p}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}\right)$

$\mu(e_1)$

$\left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, -\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}\right) = \overline{N_p}$

$$\mu\left(\frac{N_p + \overline{N_p}}{2}\right) \leq \frac{1}{2}\mu(N_p) + \frac{1}{2}\mu(\overline{N_p})$$

$$\mu(\beta_p e_1) \leq \frac{1}{2}\mu\left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, -\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}\right) + \frac{1}{2}\mu\left(\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}}\right) = 1$$

$\mu(\beta_p e_1) = \beta_p \mu(e_1)$

$$\Rightarrow \quad \mu(e_1) \leq \frac{1}{\beta_p}$$

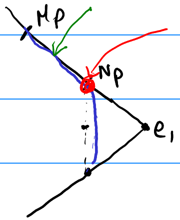
Marchand proved that $\mu(e_1) < \frac{1}{\beta_p}$

So this means $\mathcal{B}$ must go through to the blue line in the figure.

In a small region around the e_1 direction $\mu(x) > |x|_1$

But the shape could still be polygonal.

If $\mathcal{B}$ is a polygon it must have at least 8 line segments in the case where I percolate at N_p the blue curve intersects the ℓ^1 ball tangentially!



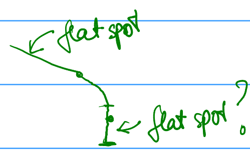
Auffinger and Damron proved that the blue curve is differentiable at N_p } → slight improve on standard method.

IMPORTANT: So the limit shape is NOT a polygon. (Combine this with the $\mu(\beta_p e_1) < 1$ observation)

Many open questions:

1) Can the limit shape contain open segments outside of the percolation cone?

2) Can there be a flat spot in the e_1 direction?



Note that this approach assumes $\tau_e = 1$ PERCOLATES. More can be said when $\tau_e \in \{1, \infty\}$ and we assume 1 percolates.

BIG OPEN QUESTION

Can we say μ is regular / differentiable / strictly convex if the weights are continuous?

But if we move away from the regime where $\{1\}$ percolate, then nothing is known.

<u>Durrett - Liggett 1981</u>

Lots of lovely ideas, and I will cover this next.