lec 5 The Cox - Durrett Limit shape theorem.

Recall $p_c$, the threshold for Bernoulli bond percolation.

holds dont form inside the growing cluster

$\mathbb{E}\left[\min_{i=1,\dots,d}\{t_i\}\right] < \infty$ for $\mu(x)<\infty$

Suppose
$$\mathbb{E}\left[\min_{i=1,\dots,2d}\{t_i\}^d\right] < \infty \quad \left(\text{a bit more than time constant condition}\right) - \#1$$
$$F(0) < p_c \quad \left(\text{no percolation of } 0\text{s} \Rightarrow \mu(e_1) \neq 0\right) \quad \#2$$

GOOD measures

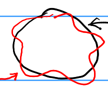
of zero.

That there is no ∞ cluster through the origin ($\mu(x)=0$)

↳ $\mu(e_1)>0 \Rightarrow \mu(x)>0 \ \forall x\neq 0$

Theorem: (Cox and Durrett)

Let $\mathcal{B} = \{x \mid \mu(x) \leq 1\}$


$\mathcal{B}$

$0<\mu(x)<\infty$ (nontrivial)

Let $B(n) = \{x \in \mathbb{R}^d \mid T(0,x) \leq n\}$

Then for any $\epsilon>0$, $\exists\ N(\omega): \Omega \to \mathbb{Z}^+$

$$P\Big(\omega \,\Big|\, \underbrace{(1-\epsilon)\mathcal{B} \subset \frac{B(n)}{n} \subset (1+\epsilon)\mathcal{B}}_{A_n} \text{ for all } n > N(\omega)\Big) = 1$$

$P(A_n) \longrightarrow 1$

$P(A_n \text{ ev}) = 1$

$= P(\cup \cap A_n) = 1$

Remark: If #2 fails the limit shape is infinite

Fix any $M$

$$P\left(\{x : x \leq M\} \subset \frac{B(n)}{n} \text{ for all large enough } n\right) = 1$$

OK, how to proceed. By the time constant theorem, one may envision the following: Pick $x \in \mathbb{Q}$ (rationals)

Then $\left\{ \lim_{n\to\infty} \frac{T(0,nx)}{n} \text{ exists}\right\} := A_x$ ← Kingman's Theorem

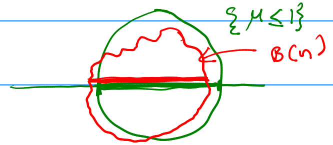
has $P(A_x) = 1$

and thus $P\left(\bigcap_{x\in\mathbb{Q}} A_x\right) = 1$ | we extended this to all $x$ in $\mathbb{R}^d$.

So for any $\epsilon$ and $x$, $\exists N_x$ st

$$\left| \frac{T(0,nx)}{n} - \mu(x) \right| < \epsilon \qquad \forall n > N_x(\omega)$$

Its easy to see the 1D projection of the shape theorem.




Let $\mathcal{B}_1 = \mathcal{B} \cap \{ \lambda e_1 \mid \lambda \in \mathbb{R} \}$

Then, its easy to see that if

$$\mathcal{B}_1(n) = \{ x = \lambda e_1 \mid T(0,x) \le n \}$$

this doesn't make a huge difference

Then with probability 1, we have

$$(1-\hat{\epsilon})\, \mathcal{B}_1 \subseteq \frac{\mathcal{B}_1(n)}{n} \subseteq (1+\hat{\epsilon})\, \mathcal{B}_1$$

This is easy to show and follows directly from

1) Existence of $\mu(e_1)$

2) Homogeneity of $\mu(\lambda e_1) = \lambda \mu(e_1)$

Fix some $x = \lambda e_1$ and

Assume $\mu(x) = 1$. Then the shape theorem gives

$(1-\epsilon)n \leq T(0, nx) \leq (1+\epsilon)n$ for large $n$ $\qquad \dfrac{T(0,nx)}{n} \longrightarrow \mu(x)$

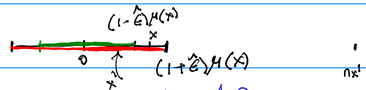
Let $y = nx + ce_1$, then $\mu(y/n) \leq 1 + \epsilon$ (continuity) $\qquad \mu\left(x + \frac{ce_1}{n}\right) \longrightarrow \mu(x)$ by continuity.

fixed $x$ and $y$

$|T(0,nx) - T(0,ny)|$
$\leq |T(nx, ny)|$
$\leq Kn|x-y|$

uniform version.

Now fix $\hat{\epsilon}$ and consider $(1+\hat{\epsilon})\mathcal{B}_1$ and $(1-\hat{\epsilon})\mathcal{B}_1$




$x' \in (1-\hat{\epsilon})\mathcal{B}_1$

Say $x' = \lambda x$ and $\mu(x') \leq 1 - \hat{\epsilon}$. Then $\forall n$ sufficiently large

all I'm saying is that $x'$ is in direction $e_1$

$\dfrac{T(0, nx')}{n} \overset{\text{Kingman}}{\leq} (1+\epsilon)\mu(x') \overset{\text{homogeneity}}{\leq} (1+\epsilon)(1-\hat{\epsilon})$
$= 1 + (\epsilon + \hat{\epsilon}) - \epsilon\hat{\epsilon}$

can arrange for this to be negative if $\epsilon \leq \dfrac{1-\hat{\epsilon}}{1+\hat{\epsilon}}$ which can certainly be done.

THUS $T(0, nx') \leq n$ and $\dfrac{\mathcal{B}_1(n)}{n}$

So certainly $x'$ is in the set $\left\{ \frac{y}{n} \,\middle|\, T(0,y) \leq n, y = \lambda x \right\}$ $\qquad (y = nx')$

Next, to show $\dfrac{\mathcal{B}_1(n)}{n} \subset (1+\hat{\epsilon})\mathcal{B}_1$

Similarly if we take a $y/n = x''$ in the set above for the sake of contradiction

Suppose $\mu(x'') > (1+\hat{\epsilon})$. Then $\dfrac{T(0, nx'')}{n} \geq (1-\epsilon)\mu(x'')$ Kingman's theorem.

$> (1-\epsilon)(1+\hat{\epsilon}) = 1 - \epsilon + \hat{\epsilon} - \epsilon\hat{\epsilon}$ #4 $\longrightarrow T(0, nx'') > n$

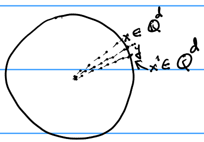
positive for small $\epsilon$, we get that #4 is certainly larger $\qquad$ ( 1 + positive qty )

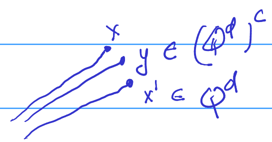
than 1, and thus $x''$ could not have belonged to the set in question

1D projection of shape theorem is obvious.

All of this choosing $n$ to be large enough can be done by considering $B_1(1-\hat{\varepsilon}) \cap \{|x|_1 > \delta\}$ on the left hand inclusion and then eventually taking $\delta$ to $0$.

OK, this can be done for 1 fixed direction for $x$ and perhaps any rational direction. But to do it for all $x \in \mathbb{R}^d$ requires some "uniformity" in our estimates.



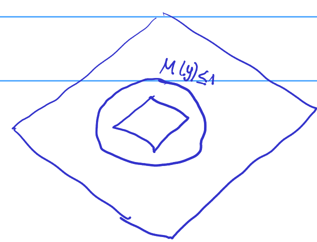

Will need good estimates on $T(x,y)$ and $T(x,y')$.

Claim: The shape theorem holding is equivalent to saying

$$\overline{\lim_{|x|_1 \to \infty}} \frac{T(0,x) - \mu(x)}{|x|_1} = 0 \qquad — (\#5)$$

Pf: The 1st step is to use a simple bound on $\mu(y)$. Since $\mu(y)$ is a nondegenerate norm of $\mathbb{R}^d$, (all finite dim norms are equivalent: — (#6)

$$C_1 |y|_1 \le \mu(y) \le C_2 |y|_1 \qquad \forall\, y \in \mathbb{R}^d$$



in the statement of the theorem.

Pf: Suppose $\boxed{(\#5)}$ holds and let $y \in \mathcal{B}(n)$ (If #5 holds then the limit shape result also holds)

For $\epsilon > 0$ $\exists$ $K$ st $|y|_1 > K$ (by #5)

$$\mu(y) - \epsilon|y|_1 \leq T(0,y) \leq \mu(y) + \epsilon|y_1| \rightarrow \overline{\lim_{|y|_1 \to \infty}} \left| \frac{T(0,y) - \mu(y)}{|y|_1} \right| = 0$$

Let $\sup_{|x|_1 \leq K} \left| T(0,x) - \mu(x) \right| = C_3$ (on $K$ and the randomness)

Take $y \in \mathcal{B}(n)$, $T(0,y) \leq n$. If $|y|_1 > K$, then (Lets show $\frac{\mathcal{B}(n)}{n} \subseteq (1+\hat{\epsilon})\mathcal{B}$

$\mu(y) - \epsilon|y|_1 \leq n$ and by #6 ($\mu$ is a norm on $\mathbb{R}^d$) $\mu(y) \geq C_1 |y_1|$

$|y|_1 \leq \frac{n}{C_1 - \epsilon} \leq C_4 n$. Thus

$$\mu(y/n) = \frac{1}{n}\mu(y) \leq \frac{n + \epsilon|y|_1}{n}$$

$$\leq \left(1 + \epsilon\frac{|y|_1}{n}\right) \leq 1 + \epsilon C$$

using the fact that $|y|_1 \leq \max(Cn, K)$

$$\frac{\mathcal{B}(n)}{n} = \left\{ \frac{y}{n} : T(0,y) \leq n \right\} \subseteq (1+\hat{\epsilon})\mathcal{B}$$

Thus $\frac{\mathcal{B}(n)}{n} \subseteq (1 + \underbrace{\epsilon C}_{\hat{\epsilon}})\mathcal{B}$

Next, show $(1-\epsilon')\mathcal{B} \subseteq \frac{\mathcal{B}(n)}{n}$ for all large $n$

Similarly if $y \in \{\mu(y) \leq 1 - \epsilon'\}$ $\quad |y|_1 \leq \frac{1-\epsilon'}{C_2}$ ($\ell_1$ norm of $|y|$ is bounded)

using the fact that $\mu$ is a norm. $1 - \epsilon' \geq \mu(y) \geq C_2|y|_1$

$\mu(ny) = n\mu(y)$ $\qquad$ $\overline{\lim}_{|x|_1 \to \infty} \dfrac{|T(0,x) - \mu(x)|}{|x|_1} \to 0$

Then by (#5) $|T(0,ny) - n\mu(y)| \leq \epsilon\, n\, |y|_1$ $\qquad$ $y \in \mathcal{B}$

Or $\quad T(0,ny) \leq n(1-\epsilon') + \underbrace{n\epsilon(1-\epsilon')}_{c_2}$ $\qquad$ $|y|_1$ is bounded.

$\leq n\left(-\epsilon' + \dfrac{\epsilon - \epsilon'\epsilon}{c}\right)$ $\qquad$ That can be made -ve by choosing $\epsilon$ to be small.

$\leq n$

and thus $\quad ny \in \mathcal{B}(n). \Rightarrow \quad y \in \dfrac{\mathcal{B}(n)}{n}$

You can thus adjust the $\epsilon$ you get from (#5) to prove

$(1-\hat{\epsilon})\mathcal{B} \subseteq \dfrac{\mathcal{B}(n)}{n} \subseteq (1+\hat{\epsilon})\mathcal{B}$

We have shown both inclusions are implied by #5.

The opposite direction is similar: that is, to prove that

$(1-\epsilon)\mathcal{B} \subset \dfrac{\mathcal{B}(n)}{n} \subset (1+\epsilon)\mathcal{B} \Rightarrow$

$\overline{\lim}_{|x|_1 \to \infty} \dfrac{T(0,x) - \mu(x)}{|x|_1} = 0$ (#5)

It's easier to prove the contrapositive; suppose

$\exists\, x_n$ st $\dfrac{T(0,x_n) - \mu(x_n)}{|x_n|_1} \to c_1 > 0$ as $|x_n| \to \infty$ (#7) (wlog). We would like $x_n$ to be rescaled so that $\frac{x_n}{n} \in (1-\varepsilon)\mathcal{B}$ and yet $T(0,x_n) > n$

This would prove $(1-\varepsilon)\mathcal{B} \not\subseteq \frac{\mathcal{B}(n)}{n}$

This is easily arranged by descending to a subsequence $\{x_{a_k}\}$ and arranging $\{b_k\}$ s.t. can be found.

$$\mu\left(\frac{x_{a_k}}{b_k}\right) = 1-\epsilon \quad \text{— (\#7a)}$$

$\mu(\lambda x_{a_k}) = \lambda \mu(x_{a_k})$

$\lambda = \frac{(1-\epsilon)}{\mu(x_{a_k})}$

$b_k$ //

Follows from homo.

Then

from assumption of the contrapositive in #7

$$T(0, x_{a_k}) \geqslant \overbrace{\mu(x_{a_k}) + \frac{C}{2}|x_{a_k}|_1}$$ for all large enough $n$ and thus

$C|x_{a_k}|_1 \geqslant \mu(x_{a_k})$

$\mu$ is a norm.

$$\geqslant (1+C)\,\mu(x_{a_k}) \quad (\#7)$$

$$= (1+C)(1-\varepsilon)\, b_k \geqslant b_k$$

(for $\epsilon$ small enough $(1+C)(1-\epsilon) \geqslant 1$)

with positive probability for small enough $\epsilon$.

So infinitely often (or for infinitely many $k$, $|x_{a_k}|_1 \to \infty$ $b_k \to \infty$)

$x_{a_k} \notin \mathcal{B}(b_k)$ (even though $\underbrace{\frac{x_{a_k}}{b_k} \in (1-\epsilon)\mathcal{B}}$) we assumed this

$\Leftrightarrow \frac{x_{a_k}}{b_k} \notin \frac{\mathcal{B}(b_k)}{b_k}$

$(1-\epsilon)\mathcal{B} \not\subseteq \frac{\mathcal{B}(n)}{n}$

Let me highlight the uniformity in this statement.

$$\overline{\lim_{|x|_1 \to \infty}} \left| \frac{T(0,x) - \mu(x)}{|x|_1} \right| = 0$$

$$= \overline{\lim_{n \to \infty}} \sup_{|x| \geq n} \frac{|T(0,x) - \mu(x)|}{|x|_1} = 0$$

satisfy #1 moment condition $E[\min_i \tau_i^d] < \infty$
#2 $F(0) < p_c$

holds for all $z$

<u>Lemma</u> Let $F$ be a GOOD measure. Then $\exists\ K < \infty$ st for ANY $x \in \mathbb{Z}^d$

$$P\Big(\underbrace{\sup_{z \neq x} \frac{T(x,z)}{|x-z|_1} < K}_{\text{GOOD}(x)}\Big) > 0 \qquad \text{\#7}$$

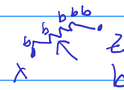
Time

$$T(x,z) < K|x-z|_1$$

Uniform constant.

We will return to this, but the argument is very similar to the idea we used to show $E[T(0,e_1)] < 1$. We also of course need the Borel Cantelli lemma.

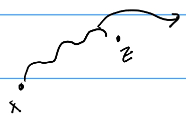
Suppose my wts are bounded. $a < \tau_e < b$




$b|x-z|_1$

The difficulty comes when $\tau$ is unbounded & we just have a moment condition #1.

What this is saying that with positive probability




$T(x,z) \leq K|x-z|$

FOR all $z$ with some uniform constant $K$ (non random)

I'll leave this in your reading list.

This gives us the "uniform" control on the passage time we desire.

Back to the proof of the shape theorem.

We argue by contradiction. Suppose the shape theorem does not hold.

Idea: Let $\{x_i\}$ be st $\frac{\overbrace{T(0,x_i) - \mu(x_i)}^{\#5}}{|x_i|_1} \to c > 0$

and since $\frac{x_i}{|x_i|_1}$ live on the unit ball, of $\ell_1$

we may descend to a subsequence and assume

$\frac{x_i}{|x_i|_1} \longrightarrow y \in \ell_1$. (Compactness of the finite dimensional unit ball)

By assumption $|T(0, x_i) - \mu(x_i)| > c|x_i|_1$

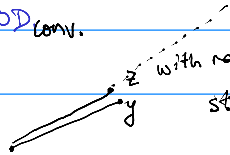
We will use the "GOOD POINTS" from the lemma to prove a contradiction to this statement.

Recall that for rational $z$, Kingman shows that

$$P\Big(\underbrace{\bigcap_{z\in\mathbb{Q}} \lim_{n\to\infty} \frac{T(0,nz)}{n} = \mu(z)}_{\text{GOOD}_{\text{conv.}}}\Big) = 1 \qquad (\#8)$$

$\text{GOOD}_{\text{conv.}}$ = the set of configurations in which convergence to the time constant $\mu(z)$ happens for all $z \in \mathbb{Q}^d$

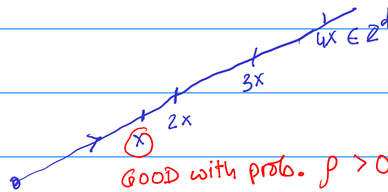
Now fix $z$ with rational coordinates.



$y$ st $|z-y| < \epsilon$

So $z$ is of the form $z = \frac{x}{M}$ where $M$ is an integer and $x \in \mathbb{Z}^d$

By continuity, $\mu(z) \approx \mu(y)$ and by the ergodic theorem (and total ergodicity)




GOOD with prob. $\rho > 0$ (a large fraction of the points are going to be good)

Infinitely many points of the form $nMz$ will be good. Then with $\tilde{n} = nM$

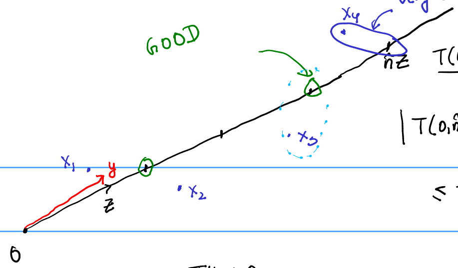
$$\left| T(0,\tilde{n}z) - T\left(0,\tilde{n}\frac{x_i}{|x_i|}\right) \right| \overset{\text{goodness.}}{\leq} T\left(\tilde{n}z, \tilde{n}\frac{x_i}{|x_i|}\right)$$
$$\leq \tilde{n}\left|z - \frac{x_i}{|x_i|}\right|$$

However $\frac{T(0,\tilde{n}z)}{\tilde{n}} \longrightarrow \mu(z) \approx \mu(y)$

and $\mu(x_i/|x_i|) \approx \mu(y)$

But $\left| \frac{T\left(0,\tilde{n}\frac{x_i}{|x_i|}\right)}{\tilde{n}} - \mu(y) \right| > c$

where $\tilde{n} = [\,|x_i|\,]$ This is a contradiction.




$\frac{T(0,\hat{n}z)}{n} \to \mu(z) \approx \mu(y)$

$\left| T(0,\hat{n}z) - T\left(0, \hat{n}\frac{x_i}{|x_i|}\right) \right|$

$\lesssim T\left(\hat{n}z, \hat{n}\frac{x_i}{|x_i|}\right) \leq n\epsilon$

THUS

$$\frac{1}{\hat{n}} T\left(0, \hat{n}\frac{x_i}{|x_i|}\right) \approx \frac{T(0,\hat{n}z)}{n} \approx \mu(z) \approx \mu(y) \approx \mu\left(\frac{x_i}{|x_i|}\right)$$

Next, we will see this in full rigor.

POLL: Do you want to see this argument with the epsilonics?

Assume $\left| \frac{T(0,x_i) - \mu(x_i)}{|x_i|} \right| \not\to 0$ equivalent to shape thm.

By compactness $\frac{x_i}{|x_i|_1} \to y$. The pts $\frac{x_i}{|x_i|_1} \in \ell^1$ ball.

$$\mu\left(\frac{x_i}{|x_i|}\right) \approx \mu(y) \approx \mu(z) \quad z \in \mathbb{Q}^d$$

$$\frac{T(0,nz)}{n} \to \mu(z)$$

But frequently $z$ is GOOD

So $\left[ \left| \frac{T(0,nz) - T\left(0, n\frac{x_i}{|x_i|}\right)}{n} \right| \right.$ is small. This is a contradiction.

Pf: Recall the equation #8 that defines the set of full measure good configurations for which

$$\frac{T(0, nu)}{n} \longrightarrow \mu(u) \quad \text{a.s } (\text{on } GOOD_{conv.})$$ for all rational $u \in \mathbb{Q}^d$

Pick $z \in \mathbb{Q}^d$ such that $|\mu(z) - \mu(y)| < \delta$, | $\mu$ is continuous

let $\xi = Mz$ s.t. $Mz \in \mathbb{Z}^d$

Claim: $\{n\xi\}_{n=1}^{\infty}$ contains a subsequence of points $\{n_k\xi\}_{k=1}^{\infty}$ such that $GOOD_{Time}(n_k\xi)$ happens

and $\frac{n_{k+1}}{n_k} \longrightarrow 1$ (happens fairly close together)

This follows from the (total) ergodicity of $T^z$

Consider $(\Omega, \mathbb{P}, T^{\xi})$ $\quad \Omega = \{w_e\}_{e \in Edges}$

$T^{\xi} w_e = w_{e+\xi}$

$$f(w) = \mathbb{1}_{\{\underbrace{GOOD_{Times}(0)}_{B_0}\}}(w)$$

$B_x = \{x \in \mathbb{Z}^d \text{ is } GOOD\}$

Ergodic average: $\frac{1}{n}\sum_{k=1}^{n} f(T^{k\xi} w) \longrightarrow \mathbb{E}[f]$ a.s

$(\Omega, \{T^z\}_{z \in \mathbb{Z}^d}, \mathbb{P})$

Ergodic: Then A is invariant if $T^z A \approx A \quad \forall z \in \mathbb{Z}^d$.

Then $\mathbb{P}(A) \in \{0,1\}$

Total ergodicity, A is inv if $T^z A \approx A$ for any $z \in \mathbb{Z}^d$

Then $\mathbb{P}(A) \in \{0,1\}$

$$\frac{1}{|B|}\sum_{x \in B} f(T^x) \to \mathbb{E}[f]$$

$\mathbb{E}[f] = P(\text{GOOD}_{\text{Times}}(0)) =: \rho > 0$ (by the lemma)

$f(T^{k\xi}\omega) = \mathbb{1}_{B_0}(T^{k\xi}\omega) = \mathbb{1}_{B_{k\xi}}(\omega)$ (indicator of the event that $k\xi$ is GOOD)

$\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{B_{k\xi}}(\omega) \longrightarrow \rho$ a.s. (ergodic)

Let $\{n_k\}_{k=1}^{\infty}$ be the random locations at which $\text{GOOD}_{\text{Times}}(n\xi)$ occurs. Thus by time $n_k$, $k$ $\text{GOOD}_{\text{Times}}$ events have occurred. (by time $n_k$ there are $k$ GOOD points)

So $\frac{k}{n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} \mathbb{1}_{B_{i\xi}} \to \rho$

So $\frac{n_{k+1}}{n_k} = \frac{n_{k+1}}{k+1} \, \frac{k+1}{k} \, \frac{k}{n_k}$

$\longrightarrow \rho \cdot \frac{1}{\rho} = 1$ ∎

$\frac{n_{k+1} - n_k}{n_k} \longrightarrow 0$

So what this is saying is that GOOD events happen fairly frequently

$(1-\epsilon)n_k \le n_{k+1} \le (1-\epsilon)n_k$

on large scales.

Assume for the sake of contradiction that (equivalent to the shape theorem)

$$\varlimsup_{|x|\to\infty} \frac{|T(0,x) - \mu(x)|}{|x|_1} > \delta \quad \text{for } \omega \in D_\delta$$

— (#9)

Take some $\omega$ in the set where:

1) $\exists \{n_k\}_{k=1}^{\infty}$ st $\text{GOOD}_{\text{Times}}(n_k z)$ happens

and $\frac{n_{k+1}}{n_k} \to 1$, $\frac{k}{n_k} \to \rho$

(this has probability one)

2) Shape theorem fails (with pos. probability)

In (#9) $\exists \{x_n\}_{n=1}^{\infty}$ st

$|T(0,x_n) - \mu(x_n)| > \delta |x_n|_1$ —

Since $\frac{x_n}{|x_n|_1} \to y$

$\left| \frac{\mu(x_n)}{|x_n|_1} - \mu(y) \right| < \frac{\delta}{10}$ for large $n$ (By continuity)

and thus

$\left| T(0, x_n) - |x_n|_1 \mu(y) \right| > \frac{\delta}{2} |x_n|_1$ "Large"

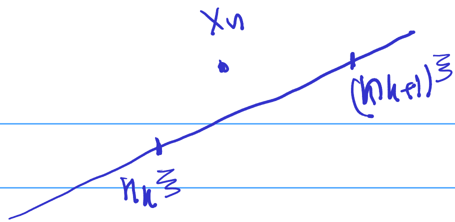
We contradict this by showing

$\left| \frac{T(0, x_n)}{|x_n|_1} - \mu(y) \right|$ is very small.

For any such $x_n$, $\exists\, k(n) \in \mathbb{Z}$ st

$$n_{k+1} M > |x_n|_1 \geq n_k M \quad \Big\} - (\#9a)$$

This is because we can choose $k(n)$ to the largest such $k$ st $n_k M \leq |x_n|_1$




(Trapping $x_n$ between GOOD points)

$$\left|\frac{T(0,x_n)}{|x_n|_1} - \mu(y)\right| \le \underbrace{\left|\frac{T(0,x_n) - T(0,n_kMz)}{|x_n|_1}\right|}_{\#10a} + \underbrace{\left|\frac{T(0,n_kMz)}{|x_n|_1} - \frac{T(0,n_kMz)}{n_kM}\right|}_{\#10b} + \underbrace{\left|\frac{T(0,n_kMz)}{n_kM} - \mu(z)\right|}_{\#10c} + \underbrace{|\mu(z) - \mu(y)|}_{\#10d}$$

$$\frac{|T(x_n, n_kMz)|}{|x_n|_1} \le K\,\frac{|n_kMz - x_n|_1}{|x_n|_1}$$

$\left(\frac{x_n}{|x_n|_1} \approx z\right)$

#10d is small by construction

#10a is small since $n_kMz$ is GOOD

#10b is small since its controlled by

$$\left|\frac{n_kMz - |x_n|_1}{|x_n|_1}\right| \left|\frac{T(0,n_kMz)}{n_kMz}\right| \qquad \frac{T(0,n_kMz)}{n_kMz} \to \mu(z)$$

follows from #9a

By construction $|x_n|_1 - n_kMz \le n_{k+1}Mz - n_kMz$

Thus $$\left|\frac{n_kMz - |x_n|_1}{|x_n|_1}\right| \le \left|\frac{n_{k+1}Mz}{n_kMz} - 1\right| \to 0 \quad \left(\text{by previous claim } \frac{n_{k+1}-n_k}{n_k} \to 0\right)$$

which is small.

I left out one part: Cox and Durrett also show that the condition is iff. If

$$\mathbb{E}\left[\min_i \tau_i^d\right] = +\infty,$$

then the "holes" we saw in the simulation form and the theorem fails to hold.

$(\Omega, \{T^z\}_{z\in\mathbb{Z}^d}, \mathbb{P})$

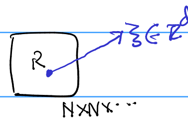
Ergodic: Then $A$ is invariant if $T^z A \approx A$ $\forall z \in \mathbb{Z}^d$. $\mathbb{P}(T^z A \,\Delta\, A) = 0$

then $P(A) \in \{0,1\}$

Total ergodicity, $A$ is inv if $T^z A \approx A$ for any $z \in \mathbb{Z}^d$

then $P(A) \in \{0,1\}$

$$\frac{1}{|B|}\sum_{x\in B} f(T^x) \to \mathbb{E}[f]$$




$$\frac{1}{|R|}\sum_{x\in R} f(T^x\omega) \longrightarrow \mathbb{E}[f]$$

$$\frac{1}{n}\sum_{i=1}^{n} f(T^{i\xi}\omega) \longrightarrow \mathbb{E}[f]$$