lec4: We want to develop easier criteria for

$X <_{var} Y$

since its hard to check

$$\int \phi \, dF_Y \le \int \phi \, dF_X \quad \forall \text{ concave increasing } \phi$$

Ex: Karlin-Novikoff (Criterion)

Let X and $\tilde{X}$ have cdfs F and $\tilde{F}$. The satisfy the cut-criterion if

$E[\tilde{X}] \le E[X]$ ←

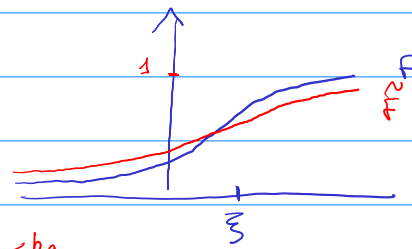
and $\exists\, \xi$ st $F(x) \le \tilde{F}(x) \quad x < \xi$

$F(x) \ge \tilde{F}(x) \quad x > \xi$

We want to demonstrate X and Y st $E[X] < E[Y]$ BUT $\mu_Y(u) < \mu_X(u)$

Compare X and Y.

If I knew this for all ϕ, then T = passage is one such fn. So then of course $T_Y(x,y) < T_X(x,y)$



Then we have $X <_{var} \tilde{X}$ $(\Rightarrow \mu_{\tilde{X}} \le \mu_X)$ (If $F \ne \tilde{F}$, $F(0) < p_c$ Then $\mu_{\tilde{X}} < \mu_X$)

van den Berg Kesten criterion.

The cut criterion implies

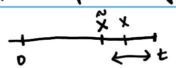
(#2) $\left[\int_{-\infty}^{x} F(x) \le \int_{-\infty}^{x} \tilde{F}(y) \quad \forall x \right.$ (when $X, \tilde{X}$ both integrable $E|X| < \infty$)

Observations:

Mean used Life: $\tilde{X}$ is smaller than X in mean used life if $E(t-\tilde{X})^+ \ge E(t-X)^+ \quad \forall t \in \mathbb{R}$

lifetime of a lightbulb

X = "lifetime of a lightbulb" say.



Queueing theory, analyzing failure rates for lightbulbs.

and the life of the lightbulb that is used is $(t-x)$.

If $\tilde{X}$ is "smaller than" X stochastically, then it uses up less of its life.

So you require $E[(t-\tilde{X})^+] \geq E[(t-X)^+]$ (assumed this)

Consider now $E[\min(t,\tilde{X})]$ and $E[\min(t,X)]$

$= E[t 1_{\{\tilde{X}>t\}}] + E[\tilde{X} 1_{\{\tilde{X}\leq t\}}]$

$= E[t(1_{\{\tilde{X}>t\}} + 1_{\{\tilde{X}\leq t\}})] + E[(\tilde{X}-t) 1_{\{t\geq\tilde{X}\}}]$

$= t - E[(t-\tilde{X})^+] \leq t - E[(t-X)^+] \quad \forall t$

$\Rightarrow \quad E[\min(t,\tilde{X})] \leq E[\min(t,X)] \quad \forall t$

is equivalent to

$E[(t-X)^+] \leq E[(t-\hat{X})^+]$ — #1

Let Ω_{cv} be the class of nondecreasing convex fns.

Recall: $\tilde{F} <_{var} F$ if

$$\int \varphi(x)\, dF \leq \int \varphi(x)\, d\tilde{F}(x) \quad \forall \varphi \in \Omega_{cv}$$

Theorem: $F <_{var} \tilde{F}$ iff $E[(t-X)^+] \leq E[(t-\tilde{X})^+]$ (for which the integrals make sense)

Let's look at $E[(t-X)^+] = -\int_{-\infty}^{t} (t-u)\, dF(u)$

$= (t-u)F(u)\Big|_{-\infty}^{t} + \int_{-\infty}^{t} F(u)\,du = \int_{-\infty}^{t} F(u)\,du$

Notice that require $\lim_{u\to-\infty} (t-u)F(u)$ to be 0

Relate this idea to Karlin-Novikoff criterion.

$t + E[(\hat{X}-t) 1_{\{t\geq\hat{X}\}}]$

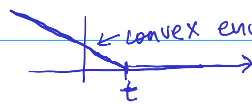
$E[(t-\hat{X})^+] \geq E[(t-X)^+] \quad \forall t$

$\Leftrightarrow E[\min(t,\hat{X})] \leq E[\min(t,X)]$

residual life of $\hat{X}$ is more. or "stochastically" $\hat{X}$ is smaller that X.

Stoyan and Daley. (pretty elementary)

$\int (t-u)^+ dF(u)$



and we can only ensure that this is true when

$$\mathbb{E}[(t-X)^+] < \infty$$

One can show that this is true when $\mathbb{E}[X^-] < \infty$

Then $\mathbb{E}[X^-] = -\int_{-\infty}^{0} u\,dF(u) = -\lim_{n\to\infty} \int_{-n}^{0} u\,dF(u)$

$$= \underbrace{-\lim_{n\to\infty} nF(-n)}\ + \lim_{n\to\infty} \int_{-\infty}^{0} F(u)\Big]\ \text{finite} \qquad \text{(by monotone converge)}$$

So this part must be 0.

When are $E(t-X)^+$ and $E(t-\hat{X})^+$ finite. Just need X and $\hat{X}$ to be integrable.

$$\int f(u)\,dF(u) = -\int f(u)\,dT(u)$$

We will prove that the cut criterion implies

$$\mathbb{E}[(t-X)^+] \leq \mathbb{E}[(t-\tilde{X})^+]$$

Pf: (of cut criterion)

Since $F(t) \leq \tilde{F}(t)$ for $t > \xi$ $\iff$ $\overbrace{\tilde{T}(t) \leq T(t)}^{\text{tail}}$

$P(X \leq t) = F(t)$, $1 - F(t) = P(X > t)$
$= T(t)$

$$\Rightarrow \int_t^\infty \tilde{T}(u)\,du \leq \int_t^\infty T(u)\,du$$

and we this.

$$\Rightarrow t + \int_t^\infty \tilde{T}(u)\,du \leq t + \int_t^\infty T(u)\,du$$

integrate by parts.

$$\Rightarrow t + \int_t^\infty u\,d\tilde{F}(u) \leq t + \int_t^\infty u\,dF(u) \qquad (\text{using } EX^+ < \infty)$$

$$\Rightarrow \mathbb{E}[\min(t, \tilde{X})] \leq \mathbb{E}[\min(t, X)] \qquad \forall\, t > \xi$$

Similarly $\mathbb{E}[\max(t, \tilde{X})] \geq \mathbb{E}[\max(t, X)] \quad \forall\, t < \xi$ $\Big]$ using a similar computation.

However $\max(t, \tilde{X}) = (\tilde{X} - t)^+ + t = \tilde{X} + t - \min(t, \tilde{X})$

$$\mathbb{E}[\max(t, \tilde{X})] = \mathbb{E}[\tilde{X}] + t - \mathbb{E}[\min(t, \tilde{X})]$$

(when $\tilde{X}$ integrable)

$$\Rightarrow \mathbb{E}[\min(t, \tilde{X})] \leq \mathbb{E}[\min(t, X)] \quad \forall\, t < \xi$$

and thus we're done.

You can take a limit $t \uparrow \xi$ and apply MCT to get the $t = \xi$ case.

<u>Tensorization</u> : What about concave functions of n variables.

$V = \mathbb{R}^n$ (a vector space in general). Let $F^{\otimes n}$ be the n-fold product measure of a cdf F on $\mathbb{R}^n$.

$f : \mathbb{R}^n \to \mathbb{R}$ is concave if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \quad \forall x, y \in \mathbb{R}^n$$

f is nondecreasing requires ordering points in $\mathbb{R}^n$: We say $x \leq y$ if $x_i \leq y_i \ \forall i = 1, \dots, n$

$$x \leq y \Rightarrow f(x) \leq f(y) \quad \text{(non decreasing)}$$

Def : Let $X = (X_1, \dots X_n)$ and $Y = (Y_1, \dots, Y_n)$ be iid vectors with $F^{\otimes n}, G^{\otimes n}$ cdfs. We say $Y <_{var} X$ if

$$\int f(x)\, dF^{\otimes n} \leq \int f(y)\, dG^{\otimes n}(y)$$

$\forall f$ integrable, nondecreasing and concave.

<u>Theorem</u> : For any n $F^{\otimes n} <_{var} G^{\otimes n}$

iff $F <_{var} G$

(see Stoyan and Daley for proof)

$F < \tilde{F}$ on $\mathbb{R}$

how about $F^{\otimes n}$ and $\tilde{F}^{\otimes n}$ on $\mathbb{R}^n$?

$T^N(x, y, w_1, \dots, w_N, d)$ and

$\tilde{T}^N(x, y, w_1, \dots w_N, d)$

Recall I needed concave AND nondecreasing in my definition of concave ordering.

One can prove this very easily for the n generalization dominance ordering

OK. Back to the vBK Kesten Example

Consider
$$\begin{cases} F_2 = \text{Unif}\{\ell, \ell+1, \dots, m\} & \ell \geq 1 \\ F_1 = F_2 * \underbrace{\text{Unif}[-\tfrac{1}{2}, \tfrac{1}{2}]}_{\text{independent} \leftarrow Y} = \text{Unif}[\ell - \tfrac{1}{2}, m + \tfrac{1}{2}] \\ F_3 = \text{Unif}[\ell, m] \end{cases}$$

rvs X_1, X_2, X_3

<u>Claim</u>: $F_2 <_{var} F_1$

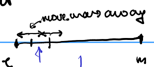
<u>Pf</u>: Fix φ convex and non decreasing.

$$\mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2+Y)] = \mathbb{E}[\mathbb{E}[\varphi(X_2+Y) \mid X_2]]$$
$$\leq \mathbb{E}[\varphi(X_2 + \mathbb{E}[Y])] = \mathbb{E}[\varphi(X_2)]$$

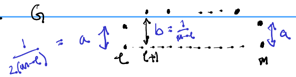
Lesson: adding an bounded mean 0 rv to an rv makes it more variable

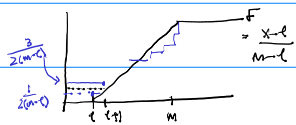
<u>Claim</u>: $F_3 <_{var} F_2$ Will show first that

$F_3 <_{var} G$



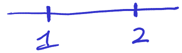
Where G





So it's clear it satisfies the cut criterion and hence

$F_3 <_{var} G$



$P(1) = \frac{1}{2}$, $P(2) = \frac{1}{2}$ $\Big\}$ X_2

$$X_1 = X_2 + Y$$

where $Y \sim \text{Unif}[-\frac{1}{2}, \frac{1}{2}]$

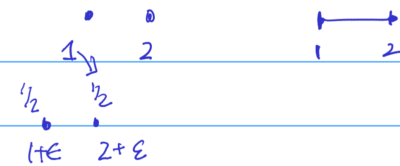
$X_1 \sim \text{Unif}[-\frac{1}{2}, \frac{5}{2}]$

$X_3 \sim \text{Unif}[1, 2]$

Repeatedly apply Karlin-Novikoff

$$F_3 <_{var} F_2 <_{var} F_1$$

$$M_{F_1} < M_{F_3} \quad \text{(vBK-Kesten)}$$



$$M^{\epsilon}_{F_2} \longrightarrow M^{\epsilon}_{F}$$

But $\mathbb{E}[M^{\epsilon}_{F_2}] = \frac{1+2+2\epsilon}{2} > \frac{1+2}{2}$

This theorem does extend to ∞ly many variables.

However its easier to simply truncate $T(x,y)$ to $[-N,N]^d$

and apply the above theorem to it.

Let $T^N(x,y) = \inf_{\substack{\gamma : x \to y \\ \gamma \subset [-N,N]^d}} T(\gamma)$

$T^N(x,y,\omega^N)$ is obviously an increasing function of ω^N (variables

in the box) and its also concave.

Thus if $G <_{var} F$ (both satisfying the condition for the time constant)

$$\mathbb{E}\, T_F^N(x,y) \leq \mathbb{E}\, T_G^N(x,y)$$

$$\to \mathbb{E}\, T_F(x,y) \leq \mathbb{E}\, T_G(x,y)$$ (apply dominated convergence)

$T^N_{(x,y)} \leq T(\Gamma_{\ell' \text{ path from } x \text{ to } y})$

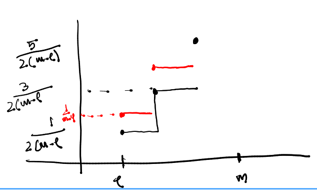
$\Rightarrow \mu_F(x) \leq \mu_G(x)$ for

any $x \in \mathbb{R}^d$

Berg Kesten says something stronger:

if F, G have finite means $G <_{var} F$ <u>and</u>

$F \neq G$ Then $\mu_F(x) < \mu_G(x)$



Thus $G <_{var} F_2$

So by their theorem

$$\mu_{F_3}(e_1) < \mu_{F_2}(e_1) < \mu_{F_1}(e_1) \qquad \text{—\#4}$$

$F_2 = \mathrm{Unif}\{[\ell - \frac{1}{2}, m + \frac{1}{2}]\}$ $\qquad \mathbb{E}[X_2] = \frac{\ell+m}{2}$

$F_2^{\epsilon} = \mathrm{Unif}\{[\ell - \frac{1}{2} + \epsilon, m + \frac{1}{2} + \epsilon]\}$ $\qquad \mathbb{E}[X_2^{\epsilon}] = \frac{\ell+m+\epsilon}{2}$

By Cox-Kesten continuity theorem

$$F_2^{\epsilon} \xrightarrow[\epsilon \downarrow 0]{d} F_2$$ (pointwise convergence at points of continuity)

$$\Rightarrow \quad \mu^{\epsilon}_{F_2}(e_1) \longrightarrow \mu_{F_2}(e_1)$$

Compare with 4 and discover $\exists\ \epsilon > 0$ st

$$\mu^{\epsilon}_{F_2}(e_1) < \mu_{F_1}(e_1)$$

But $\mathbb{E}[X_2^{\epsilon}] = \frac{\ell+m+\epsilon}{2}$, $\quad \mathbb{E}[X_1] = \frac{\ell+m}{2}$