lecture 3

Saw that $E[\min_{i=1,\dots,2d} t_i] < \infty$ is sufficient.

$$\frac{T(0,nx)}{n} \longrightarrow \mu(x) \in [0,\infty)$$

if holds then convergence a.s and in L^1

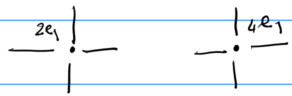
Is it necessary?

If $E[\min_{i=1,\dots,2d} t_i] = +\infty$ then for any $c>0$

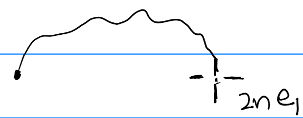
$$\sum_{n=1}^{\infty} \mathbb{P}\left(\min_{i=1,\dots,2d} t_i > cn\right) = +\infty \quad \text{— (\#1)}$$

$\longrightarrow$ $E[X] = \sum_{i=0}^{\infty} P(X > i)$ if $X \in \{0,1,2,\dots\}$

let $\{t_i^{(n)}\}_{i=1}^{2d}$ be the $2d$ edges incident to $2ne_1$



$$\left\{\frac{T(0,2ne_1)}{2n} > c \text{ i.o}\right\} \supseteq \underbrace{\left\{\min_{i=1,\dots,2d} t_i^{(2n)} > 2nc \text{ i.o.}\right\}}_{A_n} \quad \text{—(\#1a)}$$



By (#1) $\sum_{n=1}^{\infty} P(A_n) = +\infty$

2nd Borel-Cantelli says A_n occurs i.o.

since A_n are independent events.

A_n are independent events since they involve disjoint edges.

Thus $\overline{\lim_{n\to\infty}} \frac{T(0,ne_1)}{n} = +\infty$ (greater than any constant)

This condition is necessary and sufficient for $\frac{T(0,nx)}{n} \to \mu(x) \in [0,\infty)$

More facts about the time-constant that follow from SUB.

In general $\mu(e_1) \geq 0$. How big can it be?

$\mu(e_1) \leq \mathbb{E}[\tau_e]$ since

$T(0,ne_1) \leq \sum_{i=1}^{n} t_{ie_1}$ 0 - - - - - - ne_1

$\frac{1}{n}T(0,ne_1) \leq \frac{1}{n}\sum_{i=1}^{n} t_{ie_1} \to E[\tau_{e_1}]$ strong law of large #s.

If $\tau_e = 1$ identically $\mu(e_1) = \mathbb{E}[\tau_{e_1}] = 1$] equality can hold.

(When does strict ineq. hold?)

Theorem: (Hammersley and Welsh) If τ_e has at least two points of support then

$\mu(e_1) < \mathbb{E}[\tau_e]$

(They do not explicitly mention $\mathbb{E}[\min_{i=1}^{2d} \tau_i] < \infty$ but their proof needs it to make sense.

$E[\min_{i=1,\dots,2d} \tau_e] \leq \mathbb{E}[\tau_e] < \infty$

I'm not sure if it works when $E[\tau_e] = +\infty$.

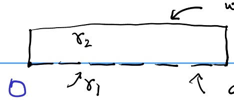
"First results" always use independent paths"

Pf: $\lim \frac{T(0,ne_1)}{n} = \inf \frac{E\,T(0,ne_1)}{n} \leq \frac{\mathbb{E}[T(0,n_0 e_1)]}{n_0}$

Choose n_0 st

So its enough to show $\frac{\mathbb{E}[T(0,n_0e_1)]}{n_0} < \mathbb{E}[\tau_e]$ — (#2a)

Pick $a<b$ st $\mathbb{P}(\tau_e \le a) > 0$ $\mathbb{P}(\tau_e \ge b) > 0$] we have assumed at least 2 points of support.



with positive probability all of those will be $\le a$ and there will be $\ge b$ } call this event A

$$n_0\mathbb{E}[\tau] = \mathbb{E}[T(\gamma_1)] = \mathbb{E}[T(\gamma_1)(1_{A^c} + 1_A)]$$
$$\geq \mathbb{E}[T(\gamma_1)1_{A^c}] + bn_0\mathbb{P}(A)$$
$$> \mathbb{E}[T(\gamma_1)1_{A^c}] + a(n_0+2)\mathbb{P}(A) \quad (\text{if } bn_0 > a(n_0+2))$$
$$\geq \mathbb{E}[T(0,n_0e_1)]$$

$T(\gamma_1) \geq T(0, n_0e_1)$

$$\mathbb{E}[T(\gamma_1)] = \mathbb{E}\sum_{i=1}^{n_0} \tau_{ie_1} = n_0\mathbb{E}[\tau_{e_1}]$$

Choose n_0 large so that $bn_0 > a(n_0+2)$

$a(n_0+2) \geq T(0,n_0e_1)$ on the event A

Divide both sides by n_0 and we're done.

Next question. When is $\mu(e_1) > 0$?

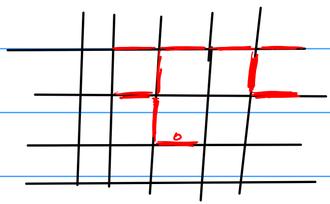
Relatively easy.

$\mu(e_1) < \mathbb{E}[\tau_{e_1}]$

$\mu(x) \le |x|_1\mathbb{E}[\tau_{e}]$

Need some background. Consider Bernoulli Bond Percolation.

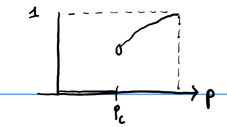
$\tau_e \in \{0,1\}$ $\mathbb{P}(\tau_e = 0) = p$



Classical question in statistical physicals.

Question: When is there an ∞ self avoiding path consisting of red edges from the origin

Let $\phi(p) = \mathbb{P}(\exists\ \infty$ self avoiding path of 0 edges from the origin$)$



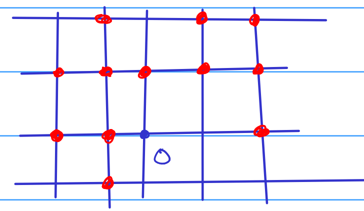
$p_c = \sup\{p : \phi(p) = 0\}$

Phase transition seen in a simple system (Emergent behavior)

H. Kesten showed that for bond percolation in $d = 2$, $p_c = \frac{1}{2}$.

What is known? $p_c \in (0,1)\ \forall d \geq 2$ (see Grimmett percolation)

Why should we care? If 0 does percolate, then $T(0, x_n) = 0$ for some $x_n \to \infty$ and this is a bad thing for us.



Theorem [Kesten, Aspects] If $F(0) < p_c$ (0 does not percolate), then $\mu(e_1) > 0$.

Pf: Lovely argument using the BK separated occurrence inequality. (☆ Good thing to read)

$\mathbb{P}(\tau_e \leq 0) < p_c$ (critical probability for bond percolation) then $\mu(e_1) > 0$.

Aspects of First-Passage Percolation (1 page calculation. Prop 5.2)

If $E[\min_{i=1,\dots,2d} \tau_i] < \infty$ and $F(0) < p_c$ ("the atom at the origin has probability lower than p_c") then $0 < \mu(e_1) < +\infty$.

What about arbitrary $x \in \mathbb{R}^d$?

We established $\frac{T(0,ne_1)}{n} \to \mu(e_1)$ a.s

So there is a bad set $A_{e_1}^c$ $P(A_{e_1}^c) = 0$ where this fails.

Good set set A_{e_1} of full meas where this happens

For any fixed x, we can similarly establish

$$\frac{T(0,nx)}{n} \longrightarrow \mu(x)$$

with a bad set A_x^c st $P(A_x^c) = 0$

Unfortunately $P(\bigcup_x A_x^c) = 0$ may not be true since this is not a countable union.

However it would be great if

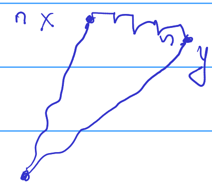
$$\frac{T(0,nx)}{n} \longrightarrow \mu(x) \quad \forall x \quad \text{a.s on}$$

some set of full measure $P(A) = 1$

<u>Standard solution:</u> Consider only rational $x \in \mathbb{Q}$

find bad sets $P(A_x^c) = 0$ and show

$$\frac{T(0,nx)}{n} \longrightarrow \mu(x) \qquad \text{a.s on } \bigcap_{x \in \mathbb{Q}} A_x$$



$$|T(0,nx) - T(0,ny)| \leq Cn|x-y|$$

Then show that μ admits a unique continuous extension to $\mathbb{R}$.

Enough to show:

<u>Prop</u>: for $x, y \in \mathbb{Q}$ $|\mu(x) - \mu(y)| \leq C|x-y|$

Pf: For any $a, b \in \mathbb{Z}^d$. Fix an ℓ^1 path between a and b. $\{a = v_0, v_1, \cdots, v_{k-1}, v_k = b\}$

$$\mathbb{E}\, T(a,b) \leq \mathbb{E} \sum_{i=0}^{k-1} T(v_i, v_{i+1}) \leq \mathbb{E}\left[\min_{i=1,\dots,2d} t_{e_i}\right] k$$

$$= \mathbb{E}\left[\min_{i=1,\dots,2d} t_{e_i}\right] |b-a|_1$$

We proved this in the previous class

$$\mathbb{E}\left[T(0,e_1)\right] \leq \mathbb{E}\left[\min_{i=1,\dots,2d} t_{e_i}\right]$$

Then $T(0,b) \leq T(0,a) + T(a,b)$

$$\mathbb{E}[T(0,b) \leq \mathbb{E}[T(0,a)] + \mathbb{E}\left[\min_{i=1,\dots,2d} t_{e_i}\right]|b-a|_1$$

set $a = nx$, $b = ny$

$$T(0,nx) \leq T(0,ny) + C\,\mathbb{E}\left[\min_{i=1,\dots,2d} t_{e_i}\right]|nx-ny|_1$$

$$\Rightarrow \quad \mu(x) \leq \mu(y) + C'|x-y|_1$$

Exchange x and y to get

$$|\mu(x) - \mu(y) \leq C'|x-y|_1$$

Then to extend μ uniquely to $\mathbb{R}^d$ is standard real analysis: Let $\{x_n\} \in \mathbb{Q}$ st $x_n \to x$, Then $\{\mu(x_n)\}$ is Cauchy and $\mu(x)$ can be defined as the limit.

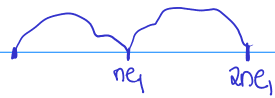
General Properties on $\mathbb{Q}$ (that extend to $\mathbb{R}$)

1) $\mu(x+y) \leq \mu(x) + \mu(y)$ Δ inequality

2) $\mu(cx) = |c|\,\mu(x)$ 1-homogeneity

3) μ is invariant under the symmetries of $\mathbb{Z}^d$ that fix the origin.

Trivial exercise



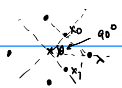
$\mu(2e_1) \leftarrow \frac{T(0,2ne_1)}{n} \cdot \frac{2}{2} \quad \frac{T(0,2ne_1)}{2n} \rightarrow \mu(e_1)$

$\mu(2e_1) = 2\mu(e_1)$ no triangle inequality. Scaling.

Proposition $\mu(x) > 0$ $\forall x \in \mathbb{R}^d \setminus \{0\}$ iff $\mu(e_1) > 0$.

Kesten's theorem helped us prove this.

Pf: If $\mu(e_1) > 0$, by 3 $\mu(e_i) > 0$ $i = 1, \dots, 2d$



If $\mu(x_0) = 0$ then so is $\mu(x_1)$

But $\mu(x_0 + x_1) = \mu(\lambda e_1) \leq 2\mu(x_0) = 0$

triangle inequality.

This is a contradiction ("proof by picture", sorry)

$\rightarrow$ $\mu(x)$ is a norm and the limit shape we see is a norm ball.

How does $\mu(\cdot)$ vary as a function of the weight cdf F?

Easy question: If $F_n \overset{d}{\Rightarrow} F$ (weak-$*$) then does $\mu_{F_n}(x) \to \mu_F(x)$ $\forall x \in \mathbb{R}^d$

If $F_n(x) \to F(x)$ at all continuity points F then $F_n \overset{d}{\Rightarrow} F$

$\mu_{F_n}(e_1) \overset{?}{\longrightarrow} \mu_F(e_1)$

Thm (2.7 in book): (Cox-Kesten 1981) If $F_n \Rightarrow F$ (both satisfying the moment bound) (A sort of domination condition)

$$\lim_{n\to\infty} \mu_{F_n}(e_1) = \mu_F(e_1)$$

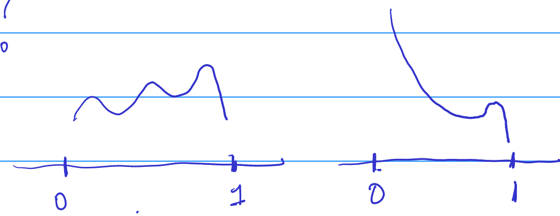
Stochastic Ordering and the vBK theorem

When can we say that

$$\mu_F(e_1) < \mu_{\tilde F}(e_1)$$

Given 2 distributions F and $\tilde F$.

When can we say one time constant is smaller than the other?



if $E[F] < E[\tilde F]$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i < \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \tilde X_i$$

First idea: (stochastic domination.)

Suppose $F, \tilde F$ are st $F(t) \le \tilde F(t)$ $\forall t$.

Then we write $F <_{DOM} \tilde F$. Let $T = 1 - F$ and $\tilde T$ similar.

Take any monotone increasing fn ϕ st $\int_{-\infty}^{\infty} |\phi| \, dF < \infty$

Then

$$\int_{\mathbb{R}} \phi(t)\, dF(t) = -\int \phi \, dT = -\phi(t)T(t)\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} \phi'(t)T(t)\,dt$$

$$\ge \int \phi'(t)\tilde T(t)\,dt.$$

$\tilde T(t) \le T(t)$

Thus $\int \phi \, dF \ge \int \phi \, d\tilde F$ — (#2)

Conversely, if (#2) holds

(By choosing $\phi =$ a we get

$$F(1) - F(a) \ge \tilde F(1) - \tilde F(a) \Rightarrow F(a) \le \tilde F(a) \ \forall a$$

So

Prop: $F <_{dom} \tilde F$ iff $\int \phi \, dF \ge \int \phi \, d\tilde F$ $\forall$ monotone increasing ϕ integrable wrt F.

This continues to hold when we take products of F and $\tilde F$.

Thm: If $F <_{dom} \tilde{F}$ (both satisfying $\mathbb{E}[\min_{i=1,\dots,2d} t_i] < \infty$)

Then $\mu_F(x) \geq \mu_{\tilde{F}}(x) \quad \forall x \in \mathbb{R}^d$

Pf: we think of $T : \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ as the passage time

where $\Omega = \{w_e\}_{e \in \text{edges}}$.

T is a monotone increasing fn in each coordinate

Thus

$$\int T(\omega,x,y)\, \overbrace{dF_1\, dF_2 \cdots}^{\text{product meas.}} = \int T(\omega,x,y)\, d\tilde{F}_1 \cdots$$

Then proceeding coordinate by coordinate completes the proof.

$\mathbb{E}[T(x,y)]$

This is not quite correct since T depends on ∞ly many rvs, but the usual analytic machinery of approximating T from above by T_n which only depends on all edges in $[-n,n]^d$ works.

Exercise: Complete this last step by yourself.

SLLN

If $F <_{dom} \tilde{F}$ and $E[F] > E[\tilde{F}]$

In $d=1$ (just the strong law)

$\mu_F(x) > \mu_{\tilde{F}}(x) \quad \forall x \in \mathbb{R}^d$

When can we ensure strict inequality?

Last exercise is for HW.

☆ Remind me to talk about ordering in next class.

concavity of $T(x,y,\omega)$

We noticed that $\mu(\lambda x+(1-\lambda)y) \leq \lambda\mu(x) + (1-\lambda)\mu(y)$

This followed from the Δ inequality

$$T(0, n[\lambda x+(1-\lambda)y]) \leq T(0, n\lambda x) + T(0, (1-\lambda)ny)$$

However note that $T(x,y,\{w_e\}_{e \in \text{Edges of } \mathbb{Z}^d})$

is CONCAVE in ω coordinates

Take $T^N(x,y)$ the truncation of $T(x,y,\omega)$ in the box $[-N,N]^d$. Clearly $T^N(x,y) \downarrow T(x,y)$.

Take the finite vector $\{\omega\}_{e \in \text{Edges } [-N,N]^d} = \vec{\omega}_n$

Then $T^N(x,y,\vec{\omega}_N) = \min\limits_{\substack{\Gamma: x\to y \\ \Gamma\subset[-N,N]^d}} T(\Gamma, \vec{\omega}_N)$

— #13

And $\min\limits_{\substack{\Gamma: x\to y \\ \Gamma\subset[-N,N]^d}} T(\Gamma, \lambda\vec{\omega}_N + (1-\lambda)\vec{\rho}_N)$

$\min\limits_{\substack{\Gamma: x\to y \\ \Gamma\subset[-N,N]^d}} \lambda T(\Gamma, \vec{\omega}_N) + (1-\lambda) T(\Gamma, \vec{\rho}_N)$

$\geq \lambda \min\limits_{\substack{\Gamma: x\to y \\ \Gamma\subset[-N,N]^d}} T(\Gamma, \vec{\omega}_N) + (1-\lambda) \min\limits_{\substack{\Gamma: x\to y \\ \Gamma\subset[-N,N]^d}} T(\Gamma, \vec{\rho}_N)$

$= \lambda T(x,y,\vec{\omega}_N) + (1-\lambda) T(x,y,\vec{\rho}_N)$

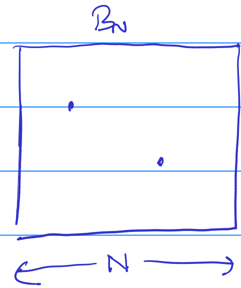
$\Rightarrow$ CONCAVE

$x \in \mathbb{R}^k$

$f(x)$ is concave if

$$f(\lambda x+(1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

$T^N(x,y,\omega)$



$\vec{\omega}_N = \{\omega_e\}_{e \text{ in } B_N}$

If I fix a path Γ from $x \to y$

Then $T(\Gamma, \lambda\vec{\omega}_N + (1-\lambda)\vec{\rho}_N)$

$= \lambda T(\Gamma, \vec{\omega}_N) + (1-\lambda) T(\Gamma, \vec{\rho}_N)$

Concave Ordering: $\overbrace{F(t) \le \tilde{F}(t)}$ is "too strong", and we want to be able to compare many pairs of distributions

extend this to more pairs of F and $\tilde{F}$.

Define an order $F <_{var} \tilde{F}$

$\Rightarrow \mu_{\tilde{F}} < \mu_F$.

$\tilde{F}$ is more variable or bigger in the concave ordering than F.

So we say $F <_{var} \tilde{F}$ is stochastically less variable if $\forall\ \phi$, concave increasing st $\int |\phi| dF < \infty$ we have $\int \phi\, dF \ge \int \phi\, d\tilde{F}$

$$\overbrace{F <_{dom} \tilde{F}}^{\text{dominance}} \Longrightarrow \overbrace{F <_{var} \tilde{F}}^{\text{concave ordering.}}$$

Intuition: If $F(t) \le \tilde{F}(t)$ the probability that $\tilde{X} \le t$ is larger that $X \le t$; ie, $\tilde{X}$ is smaller than t MORE OFTEN than X. So obviously $\mu_{\tilde{F}} \le \mu_F$. $\longrightarrow$

cdf $F, \tilde{F}$

rv $X, \tilde{X}$

Coupling.] Its a useful simple proof technique.

Similarly, if $\tilde{X}$ is more variable than X then $\tilde{X}$ is MORE LIKELY to take smaller values that we can exploit to take better paths.

Concave comparison implies that $\tilde{X}$ takes smaller values probabilistically speaking.

Ex: If $X \ge 0$ and $\tilde{X} = \min(M, X)$ $\quad \tilde{F} <_{var} F$

(trivial)

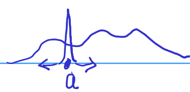
$P(X \le t) = P(\tilde{X} \le t)$

if $t < M$ if $t > M$ Then

$$\underbrace{P(\tilde{X} \le t)}_{\tilde{F}(t)} \underbrace{\ge}_{>_{dom}} \underbrace{P(X \le t)}_{F(t)}$$

Ex: Let ρ be a measure, $a \in \mathbb{R}$, $0 < p < 1$ $\epsilon > 0$

$\nu = p\,\delta_a + (1-p)\,\rho$



$\tilde{\nu} = \frac{p}{2}\delta_{a-\varepsilon} + \frac{p}{2}\delta_{a+\varepsilon} + (1-p)\rho$

$a-\varepsilon \quad a+\varepsilon$

$\int x\, d\tilde{\nu} = \int x\, d\nu$] same means.

and in an obvious way $\tilde{\nu}$ is "more variable" than ν both intuitively and by the definition of concave ordering

Theorem (van den Berg – Kesten 84 / Marchand 2002)

Suppose, $F(0) < p_c$ critical prob of percolation and $F <_{var} \tilde{F}$, then if $\tilde{F} \neq F$, $d \geq 2$

$\tilde{F}(0)$

$\tilde{\mu}(e_1) < \mu(e_1)$ (strict)

☆ In vBK, one originally needed the means to be finite.

tightness

$F_n \Rightarrow F$

$\mu_{F_n}(x) \to \mu_F$

Combining vBK-Kesten and the continuity theorem of Cox and Kesten, you can demonstrate $F, \tilde{F}$ st

$\int x\, d\tilde{F} > \int x\, dF$ but $\mu_{\tilde{F}}(e_1) < \mu_F(e_1)$

$\mathbb{E}[\tilde{X}] > \mathbb{E}[X]$

THIS IS A REMARKABLE FACT!

Compare ($d=1$ ex)

$S_n = \sum_{i=1}^n X_i$ and $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$

$\frac{S_n}{n} \to \mathbb{E}[X_1] < \mathbb{E}[\tilde{X}_1] \leftarrow \frac{\tilde{S}_n}{n}$

$d=1 \Leftrightarrow$ Strong Law of large numbers

So I will spend some time proving this

↳ ($\tilde{X}$ is stochastically smaller than X)

$\tilde{X} = f(X) = \min(M, X)$

essentially a coupling a meas. π on $\mathbb{R}\times\mathbb{R}$ that projects correctly on to F and $\tilde{F}$.

How do you say $x \leq y$ in $\mathbb{R}^n$,

$x \leq y$ if $x_i \leq y_i \;\forall i = 1, \dots, n$

In n dimensions to prove

$\mathbb{E}_F[T(x,y,\omega)] \leq \mathbb{E}_{\tilde{F}}[T(x,y,\tilde{\omega})]$

we only needed monotonicity in each coordinate, and the fact that T is increasing in each coordinate of ω.

$\nu(A) = p\,\mathbb{1}_{\{a \in A\}} + (1-p)\rho(A)$ for any meas. $A \subset \mathbb{R}$

Take any concave increasing ϕ

$$\int \phi \, d\nu = p\,\phi(a) + (1-p)\int \phi \, d\rho$$

$$\int \phi \, d\hat{\tilde{\nu}} = \frac{p}{2}\phi(a-\varepsilon) + \frac{p}{2}\phi(a+\varepsilon) + (1-p)\int \phi \, d\rho$$

Use concavity and see $\phi(a) = \phi\left(\frac{a-\varepsilon}{2} + \frac{a+\varepsilon}{2}\right) \geq \frac{1}{2}\phi(a-\varepsilon) + \frac{1}{2}\phi(a+\varepsilon)$

$\int \phi \, d\nu \geq \int \phi \, d\hat{\nu}$ $\forall$ such ϕ.

$\tilde{\nu} >_{var} \nu$