(Deift, "Universality")

lec02

The next model in the KPZ class is FPP.

let $\lambda_1(N)$ be the largest eigenvalue of a $N \times N$ Hermitian random matrix.

Then $\frac{\lambda_1(N)}{N} \longrightarrow c$ (appropriately scaled)

subtract the mean

and $\frac{\lambda_1(N) - cN}{\sigma N^{1/3}} \xrightarrow{d} F^2_{GUE}$ (Tracy Widom distribution)

$\lambda_N \le \lambda_{N-1} \cdots \le \lambda_1$

time constant

In FPP $\frac{T(0,Nx) - N g(x)}{\tilde{\sigma} N^{1/3}} \xrightarrow{d}$ conjecturally F_{GOE}

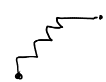
There are many physicsy arguments for the scale of fluctuations $N^{1/3}$

(Huse-Henley, Krug-Spohn) (★ good reading)

polymers — growth models like FPP & LPP

There is two example where the conjecture has been proven: Last-passage percolation with Exp(1) weights.

Here:



(0,0) (x_1,x_2) $x = (x_1, x_2) \in \mathbb{R}$

random matrices, neutron, zeta function.

$\begin{bmatrix} x_1 & x_2 & \cdots \\ \vdots & & \ddots \end{bmatrix}$ $x_{ji} = \overline{x_{ij}}$ $E[x_{ij}] = 0$ $E[x_{ij}^2] = c n^{\alpha}$

$E\frac{1}{n}\sum \lambda_i^2 = E\frac{1}{n} tr(X^2) = \frac{1}{n}\sum x_{ij}\overline{x_{ji}} = \frac{1}{n}\sum |x_{ij}|^2$

$E\frac{1}{n}\sum \lambda_i^2 = c n^{1+\alpha}$ (right scaling)

$\alpha = 0$ If expect $E[\lambda_1^2] \approx cn \Rightarrow \sqrt{E|\lambda_1|^2} \approx c\sqrt{n}$

square root

(Many simulations suggest this.)

If I chose $\alpha = 1$ then $\sqrt{E[\lambda_1^2]} = cn$

(Heuristic)

F. Rezakhanlou's notes.

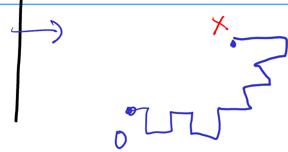
Bai and Silverstein

(Catalan #s).

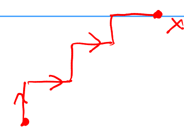
Exp(1) wts.

$\{e_1, e_2\}$ directions

FPP



LPP



Can you find a solvable case with edge weights?

Paths go "up and right", and weights are vertex weights. $\{\omega_x\}_{x\in\mathbb{Z}^2}$

$$G(x,y) = \max_{\gamma:0\to(x,y)} T(\gamma)$$

(Last-passage percolation)

where $\gamma(i)-\gamma(i-1)\in\{e_1,e_2\}$

2 solvable model (Exp and geom)

FPP

LPP (with general wts)

COMPLETELY OPEN.

$$\frac{G(N(x,y))}{N} \to g(x,y)$$

"time constant"

"Superadditivity" (Concavity)

"Subadditivity" (Convexity)

You get nearly identical expression for the discrete analog of Exponential wts. (Geometric)

Here $g(x,y) = (\sqrt{x}+\sqrt{y})^2$

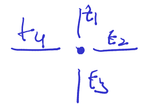
LPP $\to$ Combinatorial problem (LIS problem on S_n).

Representation theorem

And further (Johansson 2000)

$$\frac{G(0,N\vec{x}) - Ng(\vec{x})}{\sigma(\vec{x})N^{1/3}} \xrightarrow{d} F_{GUE}$$

Remark: These are all in dimension 2. We do not know what happens in $d\geq 3$.



FPP

Ch2 The time constant $\mu(x)$

Thm (Thm 2.18 in Kesten, Aspects)

Assume $\mathbb{E}\min\{t_1,\dots,t_{2d}\}<\infty$ $\quad d\geq 2$.

where $\{t_i\}$ are iid copies of t_e. $\exists\ \mu(e_1)\in[0,\infty)$

st $\lim_{n\to\infty}\frac{T(0,ne_1)}{n}=\mu(e_1)$ a.s and in L^1

random quantity

$$\frac{T(0,Nx)}{N}\to\mu(x).$$

could be ∞.

$$\mathbb{E}[\min\{t_1\dots t_{2d}\}] < \mathbb{E}[t_1]$$

$\uparrow\infty$

$P(X>t)=\frac{c}{t^{3/2}}$ then

$$\mathbb{E}[\overbrace{\min(t_1\dots t_{2d})}^{Y}]=\int_0^\infty P(Y>t)\,dt=\int_0^\infty P(X>t)^{2d}\,dt$$

$$\leq C'+c\int_1^\infty\frac{1}{t^{3/2\cdot 2d}}\,dt \qquad \tfrac{3}{2}\cdot 2d>1 \quad d>\tfrac{1}{3}$$

If $E[T(0,e_1)]<\infty$

Then Fekete's lemma says

$$\lim_{n\to\infty}\frac{E[T(0,ne_1)]}{n} \text{ exists.}$$

We have already shown that

$\lim_{n\to\infty} \mathbb{E}\left[\frac{T(0, ne_1)}{n}\right]$ exists. The above says

Analog of strong law.

this convergence is almost sure. (almost every where with respect to product measure $\mathbb{P}$)

<u>Needs</u>: (subadditive ergodic theorem) $\rightarrow$ (Kingman 68)

Ergodic Theory.

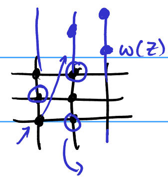
$\downarrow$ Stationarity $\searrow$ Ergodicity.

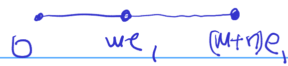
Fekete's lemma

Recall we looked at $\mathbb{E}\, T(me_1, ne_1) = a_{m,n}$

and said using "stationarity" that

$\mathbb{E}\, T(me_1, ne_1) = \mathbb{E}\, T(0, (n-m)e_1)$





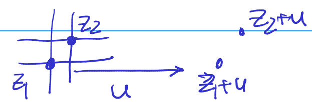
<u>Stationarity</u>: Suppose we have $\{X_z\}_{z\in\mathbb{Z}^d}$, some rvs, and suppose $\exists$ some family of meas. $\mathbb{P}_{z_1\cdots z_n}((X_{z_1}, X_{z_2}, \ldots, X_{z_n}) \in A)$

$= \mathbb{P}_{z_1+u\cdots z_n+u}(X_{z_1+u}, X_{z_2+u}, \ldots) \in A)$ $\forall$ meas. A,

n, $z_1, \ldots z_n, u \in \mathbb{Z}^d$. Then the sequence is called stationary. $\leftarrow$ #1

$\mathbb{P}_{z_1,\ldots,z_n}((X_{z_1}, X_{z_2}, \ldots, X_{z_n}) \in A)$ — Borel subset of $\mathbb{R}^d$

$= \mathbb{P}_{z_1+u,\ldots,z_n+u}((X_{z_1+u}, \ldots X_{z_n+u}) \in A)$



Let $\Omega = \mathbb{R}^{\otimes\mathbb{Z}^d}$

Then by <u>Kolmogorov extension</u> ($\leftarrow$ 406), there is $\tilde{\mathbb{P}}$,

$\tilde{\mathbb{P}} \to$ projects correctly onto this family of measures $\{\mathbb{P}_{z_1,\dots,z_n}\}_{n,z_1,\dots,z_n}$

a meas on Ω st if $M \subset \mathbb{Z}^d$ is finite

$\Pi_M : \Omega \to \mathbb{R}^{\otimes M}$ is the projection. Then for

$\Pi_M(\{\omega_z\}_{z\in\mathbb{Z}^d}) = \{\omega_z\}_{z\in M}$

any meas. $A \subset \mathbb{R}^{\otimes M}$

$$\tilde{\mathbb{P}}(\Pi^{-1}(A)) = \mathbb{P}_M((X_z)_{z\in M} \in A)$$

Single meas $\tilde{\mathbb{P}}$ agrees with this family of FD meas.

original stationary family of meas.

["original meas on stationary sequence"

Translation maps.

$\omega \in \Omega$ $\qquad \omega(z)$ wt. at z

$M^x \omega(z) = \omega(z+x)$

Define a family of commuting maps $\{M^z\}$

$M^z : \Omega \to \Omega$ as for $\omega \in \Omega$

$$M^z \omega(x) := \omega(x+z)$$

follows from the stationarity property I assumed.

Then, claim: for any A meas. (in product σ algebra)

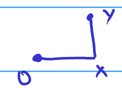
$$\tilde{\mathbb{P}}(M^z A) = \tilde{\mathbb{P}}(A)$$

probability space $\qquad$ family of commuting maps

Thus we may take this as an equivalent definition of a stationary system: $(\Omega, \mathbb{P}, \{M^z\}_{z\in\mathbb{Z}^d})$

Give me just one meas, and translate it, and every stationary seq described like this.

with commuting invertible maps M^z st $\mathbb{P}(M^z A) = \mathbb{P}(A)$ $\forall z \in \mathbb{Z}^d$



$M^x M^y = M^y M^x$

$\to$ Then my system is called stationary.

$\hookrightarrow$ I have a $\mathbb{Z}^d$ dynamical system.

Then we can extract a stationary sequence $\{X_z\} = \{\omega(z)\}_{z\in\mathbb{Z}^d}$ (If we say sequence, usually $d=1$)

Note that

$\hat{\mathbb{P}}(M^z A) = \hat{\mathbb{P}}(A)$ — #2 can be re-written | Invariance property of the meas.

$M^z A = \{ M^z \omega : \omega \in A \} = \{ \omega' : \exists\, \omega \in A \text{ st } \omega' = M^z \omega \}$

$= \{ \omega' : M^{-z} \omega' \in A \}$ $\qquad \omega \in M^z A \Leftrightarrow M^{-z} \omega \in A$

$\mathbb{1}_{M^z A}(\omega) = \mathbb{1}_A(M^{-z}\omega)$

Taking expectation and using #2 — station approximation

$\mathbb{E}[\mathbb{1}_A(M^{-z}\omega)] = \mathbb{E}[\mathbb{1}_A(\omega)]$

In general, if f is an integrable fn $f: \Omega \to \mathbb{R}$ (approximate using simple fns) and

$\mathbb{E}[f(M^z \omega)] = \mathbb{E}[f(\omega)]$ — #2 $\qquad$ Translation invariance of the meas $\hat{\mathbb{P}}$

Stationarity

$\hat{\mathbb{P}}$ is an invariant meas for $\{M^z\}_{z \in \mathbb{Z}^d}$

(Total)

<u>Ergodicity</u> The system above is called $(\Omega, \hat{\mathbb{P}}, \{M^z\})$

Ergodic if $\mathbb{P}(M^z A \,\Delta\, A) = 0$ for any z $\Rightarrow$ A is trivial $\quad (\mathbb{P}(A) \in \{0,1\})$

<u>Ex</u>: Suppose $(\Omega, \mathbb{P}, \{T^z\}_{z \in \mathbb{Z}})$ is a stationary-ergodic system. (Ergodic dynamical system)

If $f : \Omega \to \mathbb{R}$ is st $f(M^1 \omega) = f(\omega)$ a.s then f is a constant a.s.

HW

Let $f(\omega) = \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{k=1}^{n} \omega(k)$ is well defined $\leftarrow$ iid system of r.v.s.

Then $f(T\omega) = \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{k=1}^{n} \omega(k+1) = f(\omega)$

This $\Rightarrow$ $f(\omega) = C$ a.s.

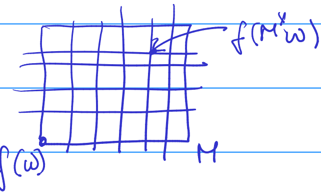
A is invariant if $\mathbb{P}(M^z A \,\Delta\, A) = 0 \quad \forall z$ $\Rightarrow$ $\mathbb{P}(A) = 0$ or 1 | Standard Ergodicity.

Further restriction on system

If we have a stationary ergodic system

(Ergodic thm): if $f : \Omega \to \mathbb{R}$ is L^1 then R is a finite box of side length N in $\mathbb{Z}^d$, $N \to \infty$

$$\frac{1}{|R|} \sum_{x \in R} f(M^x \omega) \to \mathbb{E}[f]$$



$\{X_z\}_{z \in \mathbb{Z}}$ iid. Then you think of this sequence as an element of $\Omega = \mathbb{R}^{\otimes \mathbb{Z}}$

In fact applying the ergodic theorem to $g(\omega) = \omega(0)$ gives us that

$$\frac{1}{n} \sum_{k=1}^{n} g(M^k \omega) = \frac{1}{n} \sum_{k=1}^{n} \omega(k) \to \mathbb{E}[g]$$

Theorem (Subadditive ergodic theorem) Let

$(X_{m,n})_{0 \le m < n}$ be a family of random ← "2d family"

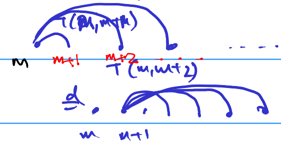
variables that satisfies the following conditions:

(SUB) 1) $X_{0,n} \le X_{0,m} + X_{m,n} \quad \forall\ 0 < m < n$

← 1) $T(0, nx) \le T(0, mx) + T(mx, nx)$

2) The sequences $(X_{m,m+k})_{k \ge 1}$ and $(X_{m+1,m+k+1})_{k \ge 1}$ have the same (finite-dim) distributions for all $m \ge 0$.

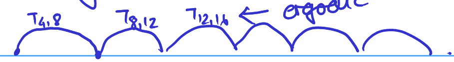
2) Stationarity.



weaker version of $\mathbb{Z}^d$ stationarity we have stated above

3) For each $k \ge 1$, the seq. $(X_{nk,(n+1)k})_{n \ge 1}$ is stationary.

3) Stationarity fix any $k=4$



4) $\mathbb{E} X_{0,1} < \infty \qquad \mathbb{E} X_{0,n} > -cn$ for some $c > 0$

4) $\mathbb{E}[T(0,x)] < \infty \qquad \mathbb{E}[T(0,nx)] \ge 0$

Then

$\lim \frac{X_{0n}}{n}$ exists a.s

$\lim_{n\to\infty} \frac{T(0,nx)}{n}$ exists.

and if the sequence in 3 is ergodic, then

$$\lim \frac{X_{0n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E} X_{0,n}}{n} = \inf_{n \ge 1} \frac{\mathbb{E} X_{0,n}}{n}$$

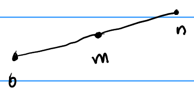
Fekete's lemma

$$\lim \frac{T(0,nx)}{n} = \lim \frac{\mathbb{E}\, T(0,nx)}{n} = \inf \frac{\mathbb{E}\, T(0,nx)}{n}$$

$X_{m,n} = T(me_1, ne_1)$ satisfies the hypothesis of the subadditive ergodic theorem

1) (SUB) is easy to obtain.

$$T(0, ne_1) \leq T(0, me_1) + T(me_1, ne_1)$$



2) $(X_{m,m+k})_{k \geq 1}$ has the same distribution as $(X_{m+1,m+1+k})_{k\geq 1}$.

[Can shift long range passage times]

$T(me_1, m+ke_1)$

$T(me_{1}, m+ke_1)$

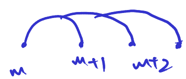
Ex: Prove this using $\mathbb{Z}^d$ stationarity that we have defined.

Same idea for both 2 and 3.

How to prove something like this? $\Omega = \{\mathbb{R}^{\mathbb{Z}^d}\}$ space of edge weights. Define

$T: \mathbb{Z}^d \times \mathbb{Z}^d \times \Omega$ $\qquad T(x, y, \omega)$ is the passage time from x to y using weights ω.

$$T(me_1, (m+k)e_1, \omega(\cdot - e_1)) \quad T((m+1)e_1, (m+k)e_1, \omega(\cdot))$$



$$T((m+1)e_1, \dots) = T(me_1, (m+1)e_1, \omega(\cdot + e_1))$$

Then, to show

$$\Big(T(me_1, (m+1)e_1, \omega), \ldots, T(me_1, (m+k)e_1, \omega)\Big) = Y_1 = (X_{m,m+1}, X_{m,m+2}, X_{m,m+3}, \ldots X_{m,m+k})$$

$$\stackrel{d}{=} \Big(T((m+1)e_1, (m+2)e_1, \omega), \ldots, T((m+1)e_1, (m+k)e_1, \omega)\Big) = Y_2 \stackrel{d}{=} (X_{m+1,m+2}, X_{m+1,m+3}, \ldots X_{m+1,m+k+1})$$

$$P(Y_1 \in A) = P(Y_2 \in A)$$

Is equivalent to showing

$$\mathbb{E}[f(\omega)] = \mathbb{E}[f(M^z \omega)]$$

$$\mathbb{E}\Big[\underbrace{1_{(Y_1 \in A)}(\omega)}_{f}\Big] = \mathbb{E}\Big[\underbrace{1_{\{Y_2 \in A\}}(\omega)}_{= f(M^{e_1}\omega)}\Big]$$

$f(\omega)$

for any (say) bounded function f.

This follows from the definition.

☆ Rest is an exercise.

3) $(X_{nk, (n+1)k})_{n \geq 1}$ for each fixed k is stationary. easy to show from our setup.

$(X_{k,2k}, X_{2k,3k}, X_{3k,4k}, \ldots)$ also

a stationary sequence. [Short range passage times]

In Fekete's lemma we used this stationarity

$$\mathbb{E}[T(me_1, (m+n)e_1)] = \mathbb{E}[T(0, ne_1)]$$

$$\mathbb{E}[X_{m,m+n}] = \mathbb{E}[X_{0,n}]$$

This corresponds to 3) in the conditions for the subadditive ergodic theorem.

☆ I don't understand why its formulated in this language in the probability literature.

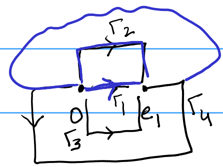
1) easy 2), 3) stationarity.

4) $\mathbb{E}[X_{0,1}] < \infty$. Where we use the condition $\mathbb{E}[\min\{t_1,\dots,t_{2d}\}] < \infty$

weaker $\mathbb{E}[t_1] < \infty$

5) So finally have to show $\mathbb{E}[\overset{X_{0,1}}{T(0,e_1)}] < \infty$

and $X_{0,n} = \mathbb{E}[T(0,ne_1)] > -cn$ (trivial)



In general we have 2d disjoint paths. $\{\Gamma_1, \dots, \Gamma_{2d}\}$

$|\Gamma_i| \le B$ (maximal length)

(In $d=2$, $B=9$) In general B can depend on d.

Then $T(0,e_1) \le \min_{i=1,..2d} T(\Gamma_i)$

$\Rightarrow \mathbb{P}(T(0,e_1) > s) \le \mathbb{P}(\min_{i=1,..2d} T(\Gamma_i) > s)$

$$\mathbb{E}[T(0,e_1)] = \int_0^\infty \mathbb{P}(T(0,e_1) > s)\, ds$$

$= \mathbb{P}(T(\Gamma_i) > s,\ i=1,\dots,2d)$ are disjoint and involve independent rvs.

$= \prod_{i=1}^{2d} \mathbb{P}(T(\Gamma_i) > s)$ (← #2)

$\mathbb{P}(\overset{\sum_{e\in\Gamma_i}\tau_e}{T(\Gamma_i)} > s) \le \mathbb{P}$(at least one edge e in $\Gamma_i = \{e_{j_1},\dots,e_{j_{k_i}}\}$ takes time $> \frac{s}{k_i}$)

If $\tau_e \le \frac{s}{B} \Rightarrow \sum \tau_e \le s$

$= \mathbb{P}(\bigcup_{u=1,\dots,k} t_{e_u} > \frac{s}{k}) \le k_i \mathbb{P}(t_{e_u} > \frac{s}{k_i})$

$k_i = |\Gamma_i|$ (# of edges in path i)

$\le B\ \mathbb{P}(t_{e_u} > \frac{s}{B})$ union bound.

$\Rightarrow$ (#2) becomes

$$\mathbb{P}(T(0,e_1) > s) \le \left[B\,\mathbb{P}(t_e > \tfrac{s}{B})\right]^{2d}$$

$$= B^{2d}\, \mathbb{P}(\min_{i=1,\dots,2d} t_{e_i} > \tfrac{s}{B})$$

Thus $\mathbb{E}[T(0,e_1)^r] = \int_0^\infty r s^{r-1} \mathbb{P}(T(0,e_1) > s)\, ds$

$\int_0^\infty \mathbb{P}(T(0,e_1) > s) = \mathbb{E}[T(0,e_1)]$ $(r=1)$

can be bounded!

$$\le \int_0^\infty r s^{r-1} B^{2d}\, \mathbb{P}(t_e > \tfrac{s}{B})^{2d} = \int_0^\infty r s^{r-1} B^{2d}\, \mathbb{P}(\min_{i=1,\dots,2d} t_{e_i} > \tfrac{s}{B})\, ds$$

$$\text{cov } t = \frac{s}{B}$$

$$= \int_0^\infty r\,(Bt)^{r-1}\, B^{2d}\, P\Big(\min_{i=1,\dots,2d} t_{e_i} > t\Big)\, B\, dt$$

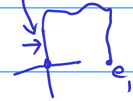
$$= B^{2d+r}\, \mathbb{E}\Big[\min_{i=1,\dots,2d} t_{e_i}^{\,r}\Big] < \infty$$

Ex:

Lemma $E\left[T(0,e_1)^r\right] < \infty$

$$\Leftrightarrow \quad \mathbb{E}\Big[\min_{i=1,\dots,2d} \{t_i\}^r\Big] < \infty$$

$\Rightarrow \quad \min_{i=1,\dots,2d} \{t_{e_i}\} \leq T(0,e_1)$



This shows that $\mathbb{E}\Big[\min_{i=1,\dots,2d} \{t_i\}\Big] < \infty$ is sufficient to prove

$\hookrightarrow \mathbb{E}\left[T(0,e_1)\right] < \infty$ (SUB)

$$0 \leq \lim_{n\to\infty} \frac{T(0,ne_1)}{n} \leq \mathbb{E}\left[T(0,e_1)\right] < \infty$$

exists and is finite.

But is it necessary?