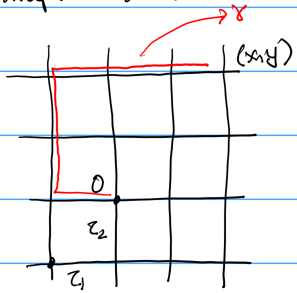
First Passage Percolation

Simple model.



$$\boxed{T(\gamma) = \sum_{e\in\gamma} \tau_e}$$

$$\underbrace{T(x,y)}_{\downarrow} = \inf_{\gamma : x\to y} T(\gamma)$$

first passage time

Extend this to $\mathbb{R}^d$.

$T : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ in the obvious way.

for any $x \in \mathbb{R}^d$ $[x]$ is the unique vertex in $\mathbb{Z}^d$

st $[x] \in [x, x+1)^d$

Limit Shape:

$B(t) = \{y \in \mathbb{R}^d : \tilde{T}(0,y) \le t\}$ (random subset of $\mathbb{R}^d$)

$E(\mathbb{Z}^d) = \{$collection of nearest neighbor edges$\}$

$\{\tau_e\}_{e\in E(\mathbb{Z}^d)} := \Omega$ is an iid set of weights.

OR vertex weights

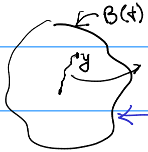
$\{\tau_x\}_{x\in\mathbb{Z}^d} =: \Omega$

$T : \mathbb{Z}^d \times \mathbb{Z}^d \times \Omega \longrightarrow \mathbb{R}$

$\tilde{T} : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}$

$\tilde{T}(x,y) = T(\underbrace{[x]}_{\downarrow}, [y])$

nearest lattice point to x



is the passage time smaller than t?

It has some sort of growth

Describe the growth of this shape.

Let F be the cdf of τ_e $\mathbb{P}(\tau_e \le x) = F(x)$.

Roydon
Durrett

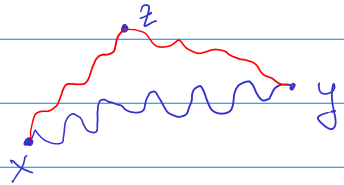
If $F(0)=0$ then $P(\tau_e \le 0) = F(0)$ (weights are nonnegative)

and $P(\tau_e = 0) = 0$

Exer: $T(x,y)$ is a metric (a.s)

1) $T(x,x)=0$

2) $T(x,y) \le T(x,z)+T(z,y)$

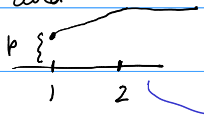


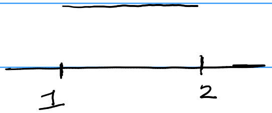
Typical question:

How does $B(t)$ behave as $t \to \infty$? (how does the "metric ball" behave)

① $\tau_e \in$ Uniform$[1,2]$ and

② τ_e has the following cdf

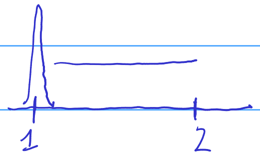




$P(\tau_e \le x)$

$= (x-1) \quad x \in [1,2]$

see the following videos on the growth of the ball. (will see ① first)

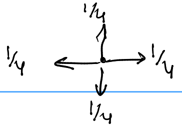


$P(\tau_e = 1) = p > \frac{1}{2}$

Guess: To first order

$$B(t) = \{(x,y) \in \mathbb{R}^2 : \sqrt{|x|^2+|y|^2} \le 1\}\, t$$

Argument for circular limit shape: take SRW in 2D.



4 directions equally likely

X_n = displacement of walk.

Pick A, nice open set.

$$P\left(\frac{X_n}{\sqrt{n}} \in A\right) \xrightarrow[CLT]{} \iint_A \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\, dx\, dy$$

Notice how the limit on the right forgets the structure of the lattice and acquires circular symmetry? (x^2+y^2 in the exponent)

(I heard Gowers mention this at some point)

This is REMARKABLE!

So we should expect something similar here.

$\{\xi_n\}_{n=1}^{\infty}$ = steps of the walk.

$\xi_n \in \{\pm e_1, \pm e_2\}$

$X_n = \xi_1 + \xi_2 + \cdots + \xi_n$

Normal density has circular symmetry.

Circular symmetry is "emergent behavior"

More refined questions

related

1) How does $T(x,y)$ behave as $|x-y| \to \infty$

2) Does $B(t)$ have a limit?

3) How do geodesics behave?

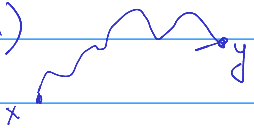
4) How does the cdf of τ_e affect all of the above?

5) Why do we care (Broader Context)

- We need basic theorems before giving this and so I'll do this at the end of class.

Question? $B(t) = tB_0 + o(t)$ with high probability.

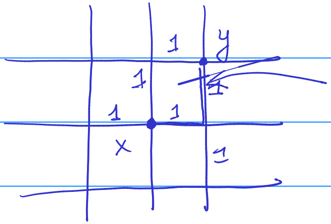
T is a metric. Between far away points (one can show there is an optimal path)



$\tau_e \in \{1,2\}$ Can you say something about B_0 (assuming it exists)

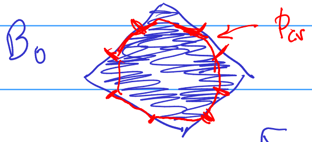
$\mathbb{P}(\tau_e = 1) = p \in (0,1)$.

If $p=1$



ℓ^1 path or shortest possible path.

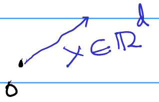
$T(x,y) = |x-y|$,



$\overbrace{\text{Time constant}}^{\text{time constant will describe } B_0}$ (First order asymptotic)

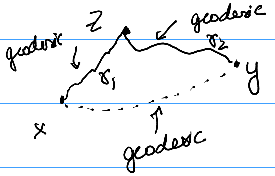
want to understand "average time to go in a particular direction"

$N(1,1) = Nx$



$\frac{T(0,Nx)}{N}$ = "average time to go in direction x"

Recall △ inequality



$$T(x,z) + T(z,y)$$
$$= T(\gamma_1) + T(\gamma_2)$$
$$= \underbrace{T(\gamma_1 : \gamma_2)}_{\text{path concatenation}}$$
$$\geq T(x,y)$$

So $T(0, (N+M)x) \leq T(0,Nx) + T(Mx,(N+M)x)$

Take expectation

$$E[T(0,(N+M)x)] \leq E[T(0,Nx)] + E[T(Nx,(N+M)x)]$$

$B(t) \approx t B_0$

But $B(t)$ is a subset.

Need some function $u(x,t)$ st $B(t)$ is a sublevel set

$$B(t) := \{x \in \mathbb{R}^d : u(x,t) \leq 1\}$$

$$\frac{T(0,Nx)}{N} \xrightarrow{?} \mu_0(x)$$

$$B_0 = \{x : \mu_0(x) \leq 1\}$$

Subadditivity in N.

→ triangle inequality

Important!

$$= E[T(0,Nx)] \overset{\downarrow}{+} E[T(0, Mx)] \quad \text{—\#1}$$

$$a_k = E[T(0,kx)]$$

$$a_{N+M} \leq a_N + a_M \quad \longleftarrow \quad \{a_k\}_{k=1}^{\infty} \text{ is a subadditive sequence}$$

Translation invariance of the measure!

Shift the origin to $[Nx]$.

Let us at least state this last result precisely.

$\Omega = \{\tau_e\}^{\mathbb{Z}^d}$, $\mathcal{F}$ = product σ algebra

$\mathbb{P} = \mathbb{P}_f^{\otimes \mathbb{Z}^d}$ ← product measure of edge wts.

Lemma If $f : \Omega \to \mathbb{R}$ $\mathbb{E}[|f|] < \infty$

Then if $f(x+\cdot) := f(\{x+z\}_{z\in\mathbb{Z}^d})$

We must have

$$\mathbb{E}[f(x+\cdot)] = \mathbb{E}[f(0+\cdot)]$$

Pf: Easy for finitely many variables.

How to do for ∞ variables?

So we have a sequence st

$$a_{N+M} \le a_N + a_M \qquad \forall\ N,M.$$

Called SUBADDITIVE!

Fekete's Lemma: Let $\{a_n\}$ be a subadditive sequence.

Then $\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n} \in [-\infty,\infty]$

Ex: $\tau_e \in \{1,2\}$ $\quad P(\tau_e = 1) = p$

In our case $0 \le \frac{a_n}{n} \le E[\tau_e]$ ← average weight

$E[\tau_e] = p + 2(1-p) = 2-p$

Pf: $a_{kn} \le \overbrace{a_n + a_n + \cdots}^{k \text{ times}} = ka_n$ ← induction $\quad a_2 = a_{1+1} \le a_1 + a_1$

$\Rightarrow \frac{a_{kn}}{kn} \le \frac{a_n}{n}$ (Hint of monotonicity)

So in general, fix m and look at n large.

We wish to show that $\frac{a_m}{m}$ is decreasing

$n = km+r$, where $r \in \{0,\dots,m-1\}$

$a_{km+r} \le ka_m + a_r$ (using subadditivity)

$$\frac{a_{km+r}}{n} \le \frac{ka_m}{n} + \frac{a_r}{n} \le \frac{a_m}{m+r/k} + \frac{a_r}{n} \rightarrow$$

m is fixed

$a_r \le \max |a_0|, \dots |a_{m-1}|$

$$\overline{\lim_{n\to\infty}}\ \frac{a_n}{n} \le \frac{a_m}{m}$$

$$\underline{\lim}_{m\to\infty}\ \overline{\lim}_{n\to\infty} \frac{a_n}{n\ n} \le \underline{\lim}_{m\to\infty} \frac{a_m}{m} \Rightarrow \frac{\overline{\lim}\, a_n}{n} \le \underline{\lim}\ \frac{a_m}{m}$$

Early results.

A generalization of classical percolation theory.

(Hammersley and Welsh 1964): for $x \in \mathbb{R}^d$

$$\frac{T(0,nx)}{n} \to g(x)$$ in probability/measure (Excellent paper to read, used ideas similar to Fekete's lemma)

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \frac{T(0,nx)}{n} - g(x) \right| > \epsilon \right) = 0$$

$$\frac{E[T(0,nx)]}{n} \to g(x)$$

average out the randomness.

(Cox & Durrett, Kesten, Richardson) $\boxed{\frac{B(t)}{t}}$ "converges"

More refined questions:

1) How to show $\frac{T(0,nx)}{n} \to g(x)$ a.s.

(Subadditive ergodic theorem)

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{T(0,nx)}{n} = g(x) \right) = 1$$

(We will build some background)

2) In the classical prob, we consider

$S_n = \sum_{i=1}^{n} X_i$, X_i iid

$\frac{S_n}{n} \to \mu$ (LLN) a.s

THEN we show

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow N(0,1)$$

Weak law $\frac{S_n}{n} \to \overset{E[X_1]}{\boxed{\mu}}$ in prob.

Strong law $\frac{S_n}{n} \to \mu$ a.s

CLT

We will not be able to prove.

for $\underbrace{\frac{T(0,nx)}{n}}$

$g(x)$ will be unknown

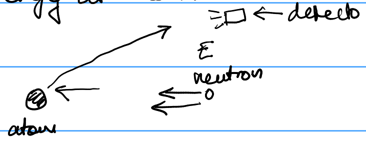
So can we show

$$\frac{T(0,nx) - ng(x)}{\sqrt{n}} \xrightarrow{d} N(0,1)$$ ← does not converge to a Normal distribution.

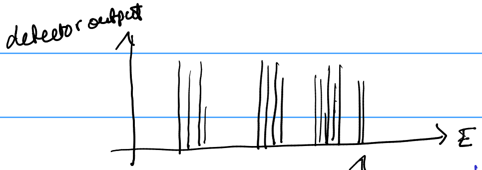
THIS TURNS OUT TO BE FALSE (simulatn)

and has an interesting history.

<u>Problem 1</u>: Take an example from Physics. A neutron scattering experiment takes one an atom and fires a bunch of neutrons at a given energy at them.



Neutrons have energy E. At most energies, neutrons pass right through, but at some E, the neutrons are SCATTERED by the atom.



↑ cannot say where exactly the resonances happen

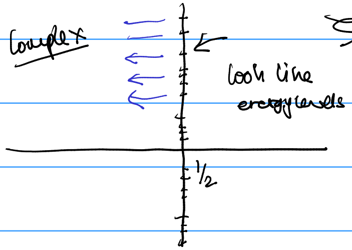
This discreteness is due to QM.

For simple atoms like hydrogen, the picture is simple, and we can predict what these energies look like. (Balmer Lines?)

But for heavy atoms (lead, Thorium, Lanthanide series) its impossible to solve QM equations

- Many, many dense but discrete resonances
- Best description is as a point process.
- Statistics (gaps, k-point correlation) are UNIVERSAL (for heavy enough atoms)

Problem 2: Zeta function



$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

s complex

Whats the distance between zeros of the zeta function?

Same statistics.

Problem 3 (Wigner Surmise)

$$\begin{bmatrix} X_{11} & \boxed{X_{12}} & \cdots & \\ & \ddots & & \\ & & \ddots & \\ & & & X_{nn} \end{bmatrix} = A \qquad A = A^* \quad (GOE) \qquad A^* = \overline{(A^T)}$$

symmetrize

$\overline{X_{ij}} = X_{ji}$

$\{X_{ij}\}$ iid $i \leq j$

$X_{ij}^r + i X_{ij}^c$

independent

X_{ij}^r, X_{ij}^c are typically $N(0,1)$

A complex, $Au = \lambda u$ column vector $\qquad A^* = \overline{(A^T)}$ complex conjugate

$\overline{(u^T A^T)} = \bar\lambda \bar u^T$ rowvector

$u^* A^* = \bar\lambda u^*$

$u^* A = \bar\lambda u^*$

$$u^* A u = u^* \lambda u = \lambda u^* u$$

$\parallel$ $\qquad$ eigenvalue eqn

$\bar\lambda u^* u$

$u^* u = \sum_{i=1}^n |u_i|^2$

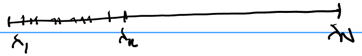
$\Rightarrow \quad \lambda = \bar\lambda$ or λ is real.

We can prove eigenvalues are real.

$$u^* u = \sum_{i=1}^n \bar u_i u_i = \sum_{i=1}^n |u_i|^2$$

$$\bar\lambda \cancel{\sum |u_i|^2} = \lambda \cancel{\sum |u_i|^2}$$

Can plot eigenvalues on a line



and can look at various statistics.

Only model in which we can prove that gap statistic are UNIVERSAL

Again Universal!

- Famous story involving Freeman Dyson and

Montgomery.

(Zeta fn results under RH)

Add

- RM results universality shown Tao-Vu,

Erdos Schlien Yau et al (2010-2015)

Original Gaussian results (1950s-1980s)

Wigner, Dyson, Mehta, Tracy, Widom, etc.