Dynamical Systems:
A Survey of the Cantor Set

Pesin and Climenhaga

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Contents

"To ask the right question is harder than to answer it."

- Georg Cantor

1 Introduction

"In most cases where a dynamical system exhibits chaotic behavior, it is associated with the presence of a fractal."[1] In this report, we explore the middle-third Cantor set predominantly; we introduce symbolic sequences as a way of representing it and also look at its topological properties. To build towards it, we first introduce dynamical systems, starting with a discussion of fractals. We see how the properties of fractals, such as self-similarity, can be expressed mathematically and how dynamical systems exhibiting chaotic behavior are closely related to them. This report is based on the content covered in lectures 1 through 3 and part of lecture 4 in chapter 1 of Y. Pesin and V. Climenhaga's 'Lectures on Fractal Geometry and Dynamical Systems.'

2 Properties of Fractals

2.1 Intricate geometry and Self-similarity

Fractals are geometric structures characterized broadly by two properties:

- intricate geometry
- self-similarity

Consider a tree in the fall season. No matter from what scale we look at it, we see the recurring branching structure, and these branches are not simple geometrical shapes such as lines or cylinders; they are much more complicated.

2.2 Infinite length and Non-differentiability

Another unusual behavior associated with fractals is infinite length. Fractal curves are also continuous everywhere and differentiable nowhere. To understand these behaviors,

[1] Lectures on Fractal Geometry and Dynamical Systems, Y. Pesin, V. Climenhaga

let us consider the example below.

2.2.1 von Koch curve

Figure 1 illustrates the procedure for constructing the von Koch curve. Notice that the Koch curve is multi-valued - a first indication that it is “unusual.”

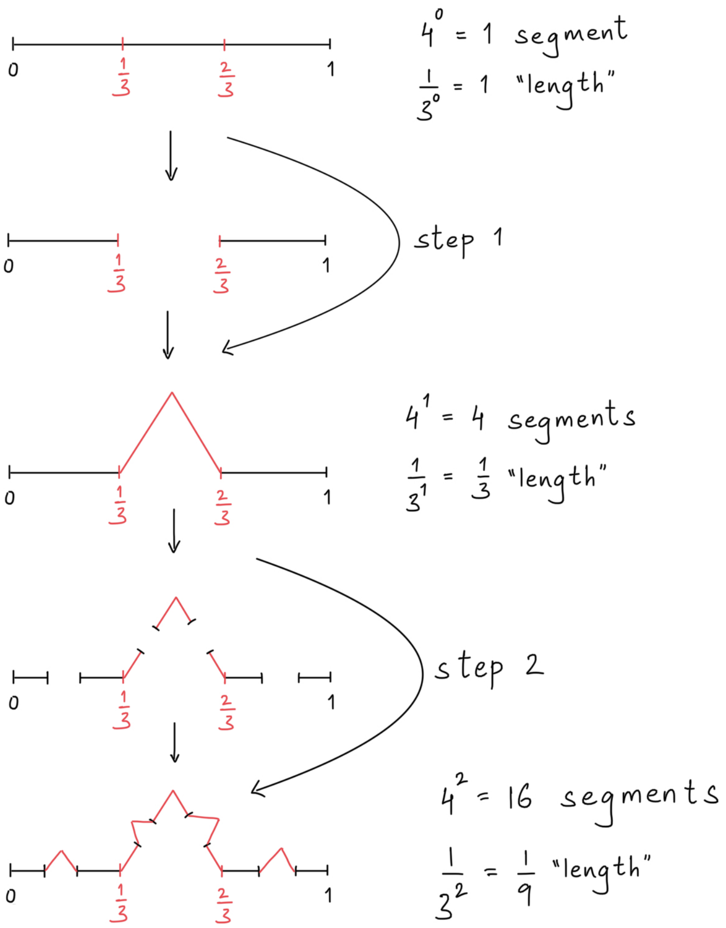

Figure 1: Procedure to construct the von Koch curve

Let us try to compute the length of the Koch curve. From figure 1 we see that the nth iteration will be a piecewise linear curve with 4^n line segments of length $\frac{1}{3^n}$. Therefore, the total length of the curve at the nth step is $\left(\frac{4}{3}\right)^n$. This quantity goes to infinity as $n \to \infty$. Thus, although the Koch curve is bounded, it has infinite length.

Since each curve in figure 1 is piecewise linear, we can parametrize them as f_1, f_2, and so on, which are maps from $[0,1] \to \mathbb{R}^2$. The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly, as shown below:

We see that

$$\sup_{x\in[0,1]}|f_{n+1}(x) - f_n(x)| \leq \frac{1}{3^n}$$

Thus, for $m < n$, we have

$$|f_n - f_m|_\infty \leq \Sigma_{k=m}^{n-1}|f_{k+1} - f_k|_\infty = \Sigma_{k=m}^{n-1}\frac{1}{3^k} \leq \frac{1}{3^m}\frac{1}{1-\frac{1}{3}}$$

Hence, the limit exists and is continuous. Although, the limit f is not differentiable anywhere.

3 Dynamics and Reiteration - a mathematical description

3.1 Discrete-time dynamical system

Consider the following map:

$$f : X \to X$$

which says $\forall\ x \in X,\ f(x) \in X$. Therefore, we can iterate f as many times without ever escaping X. We define the *range* of f as the image of the domain:

$$f(X) \coloneqq \{f(x) : x \in X\}$$

Knowing x (the initial state of the system), and the rule f that helps us determine the image of x (the immediate next state of the system), we can determine the image of $f(x)$, or $f(f(x)) = (f \circ f)(x) = f^2(x)$, and so on. In general, we see that

$$f(f^n(x)) = f^{n+1}(x)$$

and

$$f^{n+m} = f^n \circ f^m = f^m \circ f^n$$

for $n, m \in \mathbb{N}$.

The sequence of points x, $f(x)$, $f^2(x)$, ... is referred to as the *trajectory* of x. We may think of each point $x_i \in X$ (for $i = 1, 2, ..., n, n+1, ...$) specifying a particular configuration of some system. Then, f encodes a rule by which the system evolves from one state to another. It is useful to think about the number of iterations n as the amount of time elapsed, and the trajectory is the list of states that the system has passed through as time went on. This is known as the discrete-time dynamical system, as n is quantized.

This set's "identity element" is described as the case when the system is stagnant, i.e., when all x_i's are fixed.

If $f(X) = X$, then we say that the set X is *invariant*. If the domain X is invariant and f is one-to-one, then the preimage of x, $f^{-1}(x)$, is called the inverse of f and it is unique. We say that f is invertible and the map is defined to be $f^{-1} : X \to X$ (from codomain to the domain but here both are X).

In general, $f^{-n} = f^{-1} \circ f^{-1} \circ f^{-1} \circ ... \circ f^{-1}$ n times.

Similarly, the preimge of $Y \subset X$, $f^{-1}(Y) \coloneqq \{x \in X : f(x) \in Y\}$.

3.2 Exercise 1.1

3.2.1 (a) $f(x) = |2x - 1|$

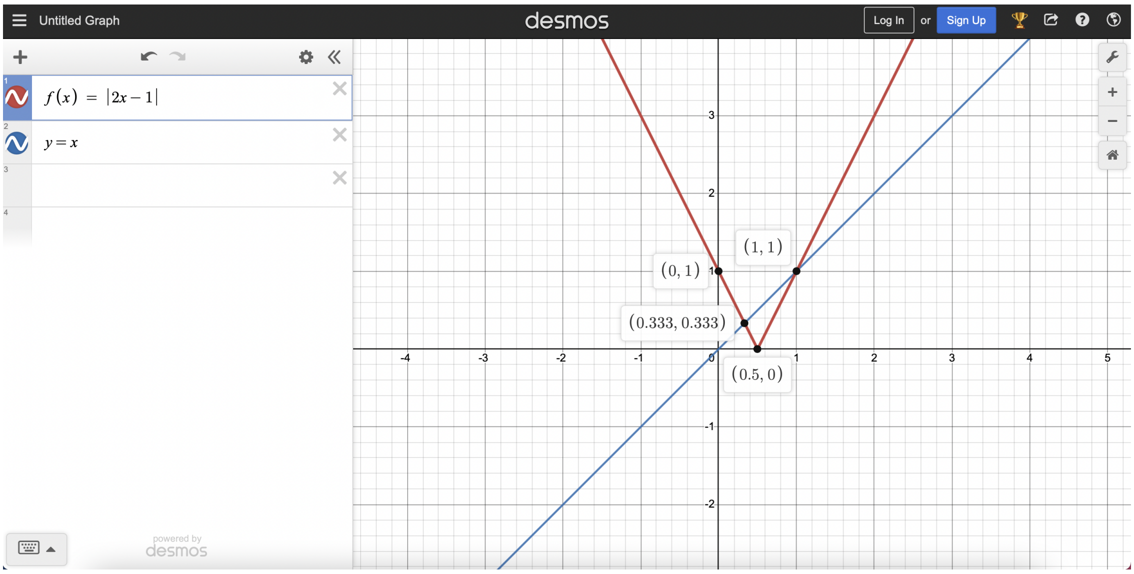

Figure 2: $f(x) = |2x - 1|$

Let us find the fixed point of $f(x)$. A fixed point of a function $f(x)$ is a point x such that $f(x) = x$ (i.e., the point(s) of the intersection of $f(x)$ and the bisectrix).

(See appendix for the procedure to determine the trajectory.)

$f(x) = x$

i.e., $|2x - 1| = x$

or, $2x - 1 = \pm x$

For $2x - 1 = x$, we have $x = 1$, which is a fixed point.

For $2x - 1 = -x$, we get $x = \frac{1}{3}$, which is also a fixed point.

Next, we divide the x-axis into 5 regions: $(-\infty, 0)$, $\left(0, \frac{1}{3}\right)$, $\left(\frac{1}{3}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 1\right)$, and $(1, \infty)$.

We pick a point x_0 in each of these intervals and see where $f(x_0)$ lies to see how the trajectory develops and what form it takes.

Let $x_0 \in (-\infty, 0)$:

For $-\infty < x_0 < 0$,

$-\infty < 2x_0 - 1 < 2(0) - 1$,

or, $\infty > |2x_0 - 1| > 1$,

i.e., $f(x_0) \in (1, \infty)$.

Note that at $x_0 = -1$, the trajectory hits the limit point 1.

Hence, the trajectory diverges.

Let $x_0 \in \left(0, \frac{1}{3}\right)$:

For $0 < x_0 < \frac{1}{3}$,

$2(0) - 1 < 2x_0 - 1 < 2\left(\frac{1}{3}\right) - 1$,

or, $1 > |2x_0 - 1| > \frac{1}{3}$,

i.e., $f(x_0) \in \left(\frac{1}{3}, 1\right)$.

Note, that at a point very close to $x = \frac{1}{3}$, the trajectory hits the limit point $\frac{1}{3}$. I was unable to find all the periodic points.

Hence, the point x_0 keeps reiterating infinitely keeping the trajectory confined within the interval $\left(\frac{1}{3}, 1\right)$.

Let $x_0 \in \left(\frac{1}{3}, \frac{1}{2}\right)$:
For $\frac{1}{3} < x_0 < \frac{1}{2}$,
$2\left(\frac{1}{3}\right) - 1 < 2x_0 - 1 < 2\left(\frac{1}{2}\right) - 1$,
or, $\frac{1}{3} > |2x_0 - 1| > 0$,
i.e., $f(x_0) \in \left(0, \frac{1}{3}\right)$.

Hence, the point x_0 keeps reiterating infinitely keeping the trajectory confined within the interval $\left(0, \frac{1}{3}\right)$.

Let $x_0 \in \left(\frac{1}{2}, 1\right)$:
For $\frac{1}{2} < x_0 < 1$,
$0 < 2x_0 - 1 < 1$,
or, $0 < |2x_0 - 1| < 1$,
i.e., $f(x_0) \in (0, 1)$.

Let us examine the values that x_0 must take for:

(a) $-(2x_0 - 1) > \frac{1}{2}$, $\left(\text{where } x_0 \in \left(0, \frac{1}{3}\right)\right)$:
$\implies x_0 < \frac{1}{4}$
Thus, $\left(0, \frac{1}{3}\right) = \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right)$

(b) $(2x_0 - 1) > \frac{1}{2}$, $\left(\text{where } x_0 \in \left(\frac{1}{2}, 1\right)\right)$:
$\implies x_0 > \frac{3}{4}$
Thus, $\left(\frac{1}{2}, 1\right) = \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right)$

Hence, $(0, 1) = \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right)$.

For $x_0 \in \left(0, \frac{1}{4}\right), f(x_0) \in \left(\frac{1}{2}, 1\right)$.
For $x_0 \in \left(\frac{1}{4}, \frac{1}{3}\right), f(x_0) \in \left(\frac{1}{3}, 1\right)$.
For $x_0 \in \left(\frac{1}{3}, \frac{1}{2}\right), f(x_0) \in \left(0, \frac{1}{3}\right)$.
For $x_0 \in \left(\frac{1}{2}, \frac{3}{4}\right), f(x_0) \in \left(0, \frac{1}{2}\right)$.

For $x_0 \in \left(\frac{3}{4}, 1\right), f(x_0) \in \left(\frac{1}{2}, 1\right)$.

$\therefore$ $x_0 \in (0,1)$ keeps $f(x_0) \in (0,1)$. Hence, the set of values x must take is:

$$\{x : |2x-1| < 1\} = \left\{x > \frac{1}{2} : (2x-1) < 1\right\} \cup \left\{x \leq \frac{1}{2} : -(2x-1) < 1\right\}$$

Let $x_0 \in (1, \infty)$:

For $1 < x_0 < \infty$,

$1 < 2x_0 - 1 < \infty$,

or, $1 < |2x_0 - 1| < \infty$,

i.e., $f(x_0) \in (1, \infty)$.

Hence, the trajectory diverges.

3.2.2 (b) $g(x) = |x-2|$

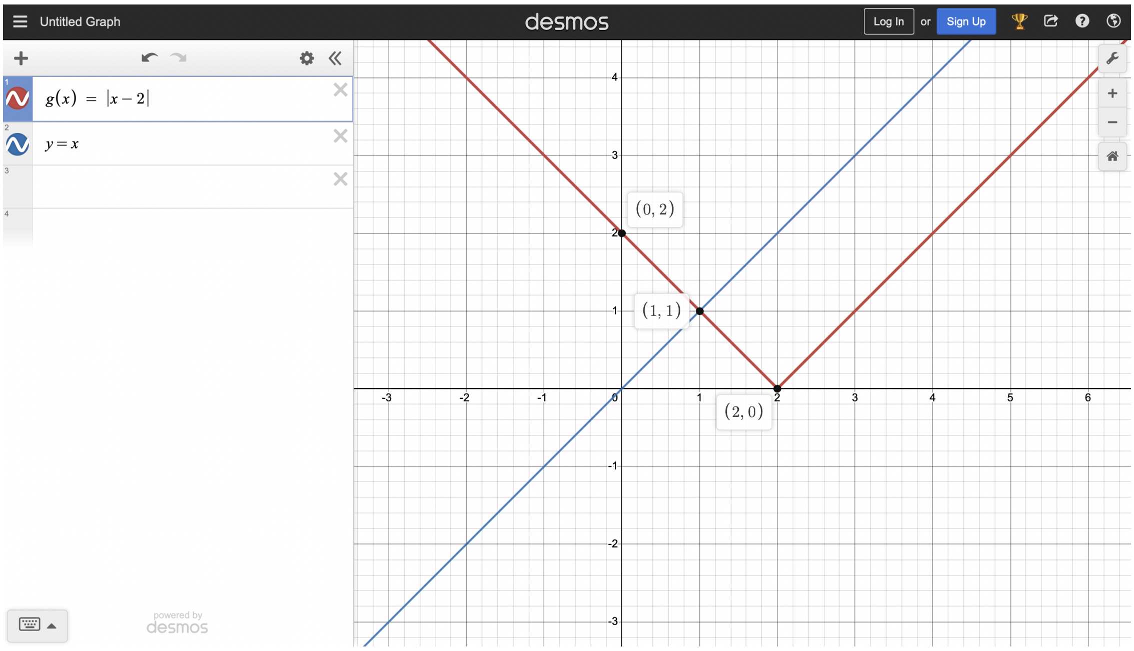

Figure 3: $g(x) = |x-2|$

Let us find the fixed point of $g(x)$. A fixed point of a function $g(x)$ is a point x such that $g(x) = x$.

$g(x) = x$

i.e., $|x - 2| = x$

or, $x - 2 = \pm x$

For $x - 2 = x$, we have $0 = 2$, which is absurd.

For $x - 2 = -x$, we get $x = 1$, which is our fixed point.

Next, we divide the x-axis into 4 regions: $(-\infty, 0), (0, 1), (1, 2)$, and $(2, \infty)$.

We pick a point x_0 in each of these intervals and see where $g(x_0)$ lies to see how the trajectory develops and what form it takes.

Let $x_0 \in (-\infty, 0)$:

For $-\infty < x_0 < 0$,

$-\infty < x_0 - 2 < 0 - 2$,

or, $\infty > |x_0 - 2| > 2$

i.e., $g(x_0) \in (2, \infty)$.

Note that the trajectory hits the limit point at $x_0 = -1$.

In this interval, $f(x) = 2 - x$. So, $f(f(x)) = f(2 - x) = 2 - (2 - x) = x$. $f(f(f(x))) = f(x) = 2 - x$. $f(f(f(f(x)))) = f(2 - x) = x$. As we see, this "oscillatory" behavior keeps on going. Thus, the trajectory is confined in (0,2).

Let $x_0 \in (0, 1)$:

For $0 < x_0 < 1$,

$0 - 2 < x_0 - 2 < 1 - 2$,

or, $2 > |x_0 - 2| > 1$

i.e. $g(x_0) \in (1, 2)$.

Let $x_0 \in (1, 2)$:

For $1 < x_0 < 2$,

$1 - 2 < x_0 - 2 < 2 - 2$,

or, $1 > |x_0 - 2| > 0$

i.e. $g(x_0) \in (0, 1)$.

Hence, for $x_0 \in (0, 2)$, $g(x_0) \in (0, 2)$ and so we see a looping behavior. Notice that every point in (0, 2) is a point with period 2; $f^2(x) = 2-(2-x) = x$ [$f(x) = 2-x$ for $x \in (0, 2)$]. Furthermore, the trajectories revolve around (1, 1), as 1 is a fixed point and acts as a 'basin.'

Let $x_0 \in (2, \infty)$:
For $2 < x_0 < \infty$,
$2 - 2 < x_0 - 2 < \infty$,
or, $0 < |x_0 - 2| < \infty$
i.e. $g(x_0) \in (0, \infty)$.

Let $f(x) \in (2, \infty)$. Then, $f^2(x) = f(f(x)) = f(x - 2) = (x - 2) - 2 = x - 4$.
Similarly, $f^3(x) = f(f(f(x))) = f(f(x - 2)) = f(x - 4) = (x - 4) - 2 = x - 6$.

Hence, if for $i = 1, 2, ..., k - 1$, $f^i(x) \in (2, \infty)$, then $f^k(x) = x - 2k$.

The trajectory is confined in (0,2).

Thus, we have a periodic orbit for $x_0 \in (0, 1) \cup (1, 2)$. For $x_0 \in (-\infty, 0) \cup (2, \infty)$, the orbit returns to the periodic orbit region which is that for $0 < x < 2$.

3.3 The coding of the trajectory of x - a paradox

Define the following continuous map:

$$f : A \to A$$

where $A \subset \mathbb{R}^2$. This restricts the entire trajectory of x to A.

Suppose we know two things: the explicit map f and the initial point x. Then, we can exactly compute each point $f^n(x)$, $n \in \mathbb{N}$ in the trajectory of x. The map f seems entirely deterministic.

Now, suppose we divide A into two subdomains A_1 and A_2 such that $A_1 \cup A_2 = A$; i.e. every $x \in A$ lies in exactly one of the two subdomains. We now focus only on in which of the two subdomains does f^n lie for $n = 1, 2, ...$ In this way, we assign each iterate a number 1, if it lies in A_1 or 2, if it lies in A_2 and form a coding of the trajectory of x as a sequence of 1's and 2's. As an example, let $x \in A_1$, $f(x) \in A_2$, $f^2(x) \in A_2$, $f^3(x) \in A_1$, and so on. This trajectory will be coded as 1221... But this means the trajectory of x will randomly jump between A_1 and A_2, and this implies that even if we know in which subdomain $x, f(x), f^2(x), ...$ lie, we cannot determine where $f^{n+1}(x)$ will lie. We can only provide a probabilistic description of this phenomenon. This means, then, that f is *not* entirely deterministic! A paradox! It is our aim to resolve this paradox in a future study and learn how and why such behavior arises from dynamical systems associated with fractal sets.

4 Population models

We find that simple population models such as $f(x) = rx$ where x is the initial population and $r > 1 \in \mathbb{R}$, give rise to trajectories that diverge; 'a small population will grow to be arbitrarily large' quickly (or for $r < 1$, the population converges to 0; all 'members die quickly'). These models are, however, unrealistic.

Hence, a revised model is introduced:

$$f : [0, 1] \to [0, 1]$$

so that

$$f(x) = rx(1 - x) \tag{1}$$

4.1 Conjugacy

We observe that f defined in (1) is equivalent to $g(x) = x^2 + c$. That is, we can find the value of $c \in \mathbb{R}$, some interval $I \subset \mathbb{R}$ and a change of coordinates $h : [0, 1] \to I$ so that

$$g(h(x)) = h(f(x)) \tag{2}$$

(2) says, if we first apply h, we go from $[0,1]$ to I. Then applying g takes us from I to I; and if we first apply f, we go from $[0,1]$ to $[0,1]$. Then applying h takes us from $[0,1]$ to I. Either way, we end up at the same point. Hence, using h (conjugacy), the dynamics of f can be studied using the dynamics of g (conjugates). Let us work out an example.

4.2 Exercise 1.2 - Explicit change of coordinates

Let $h(x) = ax + b$. We have $g(x) = x^2 + c$ and $f(x) = rx(1-x)$.

Now,
$g(h(x)) = h(f(x))$
$\therefore g(ax+b) = h(rx - rx^2)$
$\therefore (ax+b)^2 + c = a(rx - rx^2) + b$
$\therefore a^2x^2 + 2abx + b^2 + c = -arx^2 + arx + b$

Comparing the coefficients of x^2 and x and the constant terms on both sides, we have that

$a^2 = -ar \implies a = -r; (a \neq 0)$
$2ab = ar \implies b = \frac{r}{2}$
$b^2 + c = b \implies c = \frac{r}{2} - \frac{r^2}{4}$

Hence, $h(x) = -rx + \frac{r}{2}$.

We note that $f : [0,1] \to [0,1]$. For any $r \neq 0$, the maximum of f is at $x = \frac{1}{2}$. Thus $f\left(\frac{1}{2}\right) = \frac{r}{4}$, which tells us that $0 \leq r \leq 4$.

Let us examine the interval $[0,4]$ as $[0,2) \cup [2,4]$:

For $r \in [0,2)$, $c \in \left[0, \frac{1}{4}\right)$.
For $r \in [2,4], c \in [-2,0]$.

Thus, for $r \in [0,4]$, $c \in \left[-2, \frac{1}{4}\right]$.

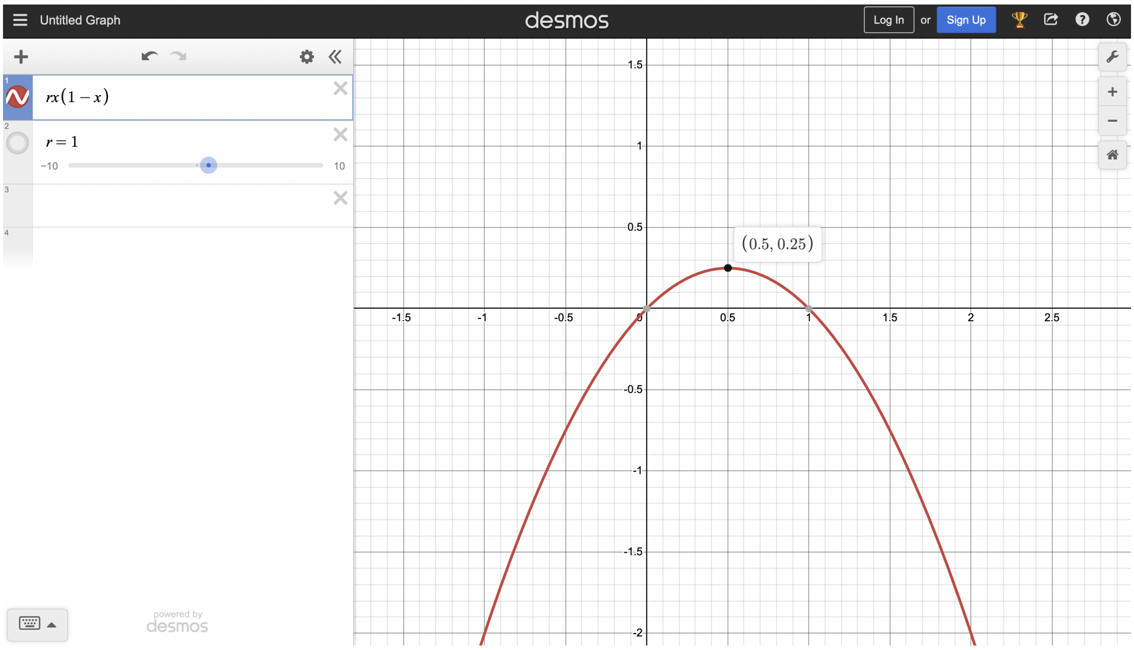

Figure 4: $f(x) = rx(1-x)$

4.3 Exercise 1.3 - Behavior of trajectories

Let us have 3 cases for what values c can take:

Case 1: $c = 0$

As evident from figure 4 above, the parabola intersects the bisectrix (the line $y = x$) at the points $(0, 0)$ and $(1, 1)$.

For any point $x_0 \in (0, 1)$, the trajectory converges to $(0, 0)$, which means the origin acts as an attracting point.

Points 0 and 1 are also the fixed points for $g(x) = x^2$. Thus, for these points, the trajectory never moves.

For the point $x_0 > 1$, the trajectory converges to the point $(1, 1)$ and so the point $(1, 1)$ acts as an attracting point.

For a point $x_0 < 0$, the trajectory diverges to infinity.

Case 2: $c = 0.25$

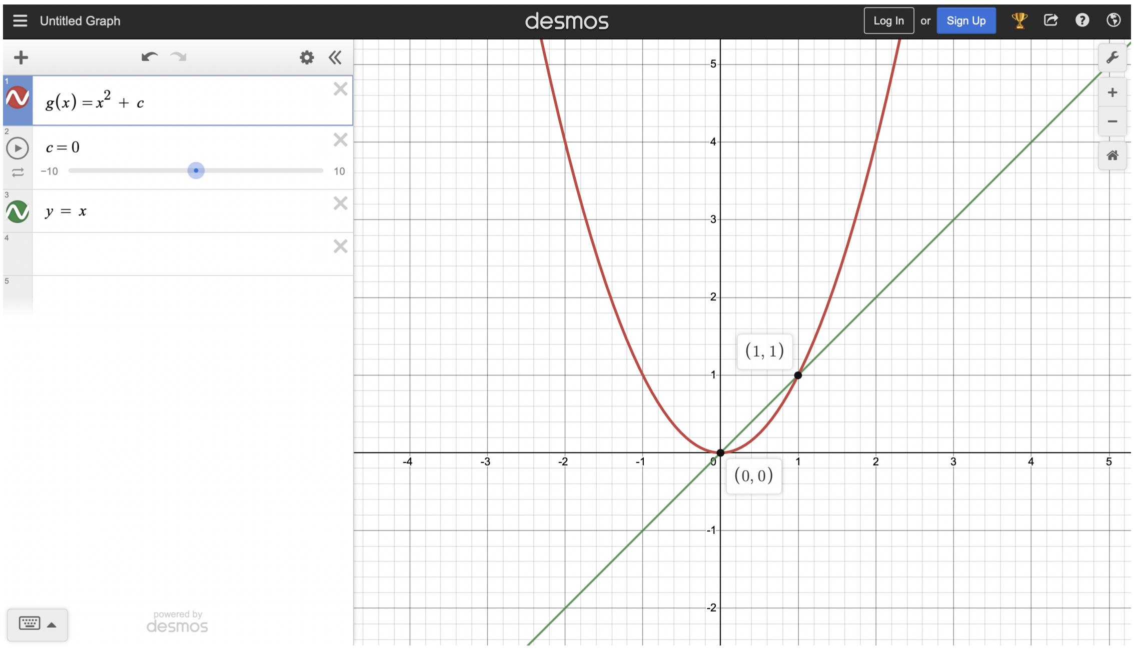

Figure 5: $g(x) = x^2$

At this value of c, $g(x)$ is tangent to the bisectrix. The bisectrix intersects $g(x)$ at the point $(0.5, 0.5)$ as evident from figure 5.

For any point $x_0 \in [0, 0.5)$, the trajectory converges to the point $(0.5, 0.5)$. Hence, $(0.5, 0.5)$ is attracting, for this case.

At $x_0 = 0.5$, the trajectory never moves, as 0.5 is a fixed point.

For $x_0 > 0.5$, the trajectory moves away from the point $(0.5, 0.5)$ and goes to infinity. Thus, $(0.5, 0.5)$ is a repelling point for this case.

For $x_0 \in (-0.5, 0]$, the trajectory converges to the point $(0.5, 0.5)$. Hence, $(0.5, 0.5)$ is an attracting point for this case as well.

For $x_0 \in (-\infty, -0.5)$, the trajectory diverges from the point $(0.5, 0.5)$. Hence, $(0.5, 0.5)$ is a repelling point for this case as well.

In conclusion, the point $(0.5, 0.5)$ is neither attracting nor repelling.

Case 3: $c \to \infty$

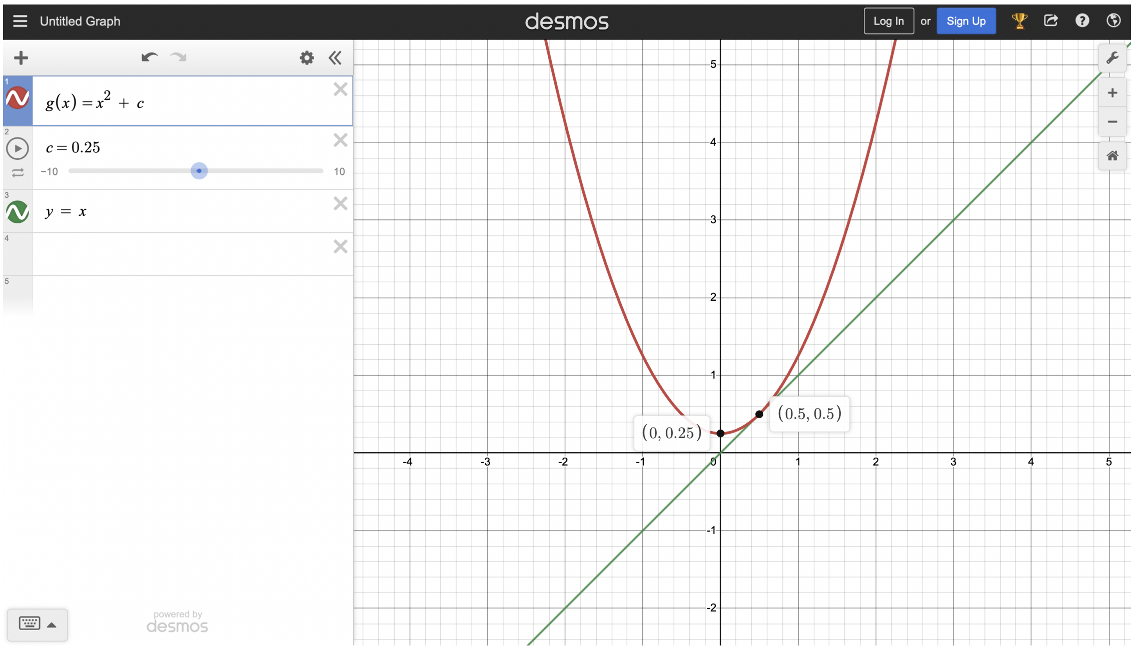

Figure 6: $g(x) = x^2 + 0.25$

For all x_0 i.e. $x_0 = 0, x_0 > 0$, and $x_0 < 0$ the trajectories escape to infinity.

5 Towards the Symbolic Sequences

We examined the complex behavior of trajectories of the map $g(x) = x^2 + c$ in the previous section. However, this complex behavior is attributed to the non-linearity of the map, which also makes it difficult to explore further. Hence, we look at piecewise linear maps in this section, which are tractable, unlike non-linear maps, and whose trajectories exhibit chaotic behavior.

Consider the map

$$f : D = I_1 \cup I_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \to [0, 1]$$

defined to be piecewise linear on the intervals $I_1 = \left[0, \frac{1}{3}\right]$ and $I_2 = \left[\frac{2}{3}, 1\right]$, and such that $f(I_1) = f(I_2) = [0, 1]$, shown below. Here, $[0, 1]$ is also the range of the function;

$$R = f(D) = \{f(x) : x \in D\}$$

Notice, for any $x_0 \in D$, $f(x_0) \notin D$. So, f cannot be iterated further. Thus, our first task is to construct a domain on which f^2 is defined.

For f^2 to be defined on a domain, both x and $f(x)$ must lie in the domain of f. That is, we must have $x_0 \in D \cap f^{-1}(D)$, where $f^{-1}(D) = \{x \in D : f(x) \in D\}$ and so

$$x_0 \in D \cap f^{-1}(D) = \{x : x \in D \text{ and } f(x) \in D\}$$

We find that the domain on which f^2 is defined consists of 4 intervals, $I_{11} = \left[0, \frac{1}{9}\right]$, $I_{21} = \left[\frac{2}{3}, \frac{7}{9}\right]$, $I_{12} = \left[\frac{2}{9}, \frac{1}{3}\right]$, and $I_{22} = \left[\frac{8}{9}, 1\right]$.

We see that:

$f(I_{11}) = f\left(\left[0, \frac{1}{9}\right]\right) = I_1$; $f^2(I_{11}) = f(I_1) = f\left(\left[0, \frac{1}{3}\right]\right) = [0, 1]$.

$f(I_{12}) = f\left(\left[\frac{2}{9}, \frac{1}{3}\right]\right) = I_2$; $f^2(I_{12}) = f(I_2) = f\left(\left[\frac{2}{3}, 1\right]\right) = [0, 1]$.

$f(I_{21}) = f\left(\left[\frac{2}{3}, \frac{7}{9}\right]\right) = I_1$; $f^2(I_{21}) = f(I_1) = [0, 1]$.

$f(I_{22}) = f\left(\left[\frac{8}{9}, 1\right]\right) = I_2$; $f^2(I_{21}) = [0, 1]$.

Observe that each interval $I_{w_1 w_2}$ where $w_1, w_2 = 1, 2$ can be written as

$$I_{w_1 w_2} = I_{w_1} \cap f^{-1}(I_{w_2}) \tag{3}$$

Let us verify it:

For the interval I_{11}, $w_1 = 1, w_2 = 1$.

$I_1 \cap f^{-1}(I_1) = \left[0, \frac{1}{3}\right] \cap \left[0, \frac{1}{9}\right] = \left[0, \frac{1}{9}\right] = I_{11}$

For the interval I_{12}, $w_1 = 1, w_2 = 2$.

$I_1 \cap f^{-1}(I_2) = \left[0, \frac{1}{3}\right] \cap \left[\frac{2}{9}, \frac{1}{3}\right] = \left[\frac{2}{9}, \frac{1}{3}\right] = I_{12}$

For the interval I_{21}, $w_1 = 2, w_2 = 1$.

$I_2 \cap f^{-1}(I_1) = \left[\frac{2}{3}, 1\right] \cap \left[\frac{2}{3}, \frac{7}{9}\right] = \left[\frac{2}{3}, \frac{7}{9}\right] = I_{21}$

For the interval I_{22}, $w_1 = 2, w_2 = 2$.

$I_2 \cap f^{-1}(I_2) = \left[\frac{2}{3}, 1\right] \cap \left[\frac{8}{9}, 1\right] = \left[\frac{8}{9}, 1\right] = I_{22}$

Let us summarize our process thus far:

To find the domain of f, we removed the middle-third from $[0,1]$ leaving $2^1 = 2$ intervals of length $\frac{1}{3^1} = \frac{1}{3}$. To find the domain of f^2, we removed the middle-third of each of the two intervals earlier, leaving $2^2 = 4$ intervals of length $\frac{1}{3^2} = \frac{1}{9}$ each. We encounter the same problem as before: f^2 cannot be iterated further. And so, to find the domain of f^3, we follow the pattern and recognize that we will have to remove the middle-third of each of the four intervals earlier to be left with $2^3 = 8$ intervals (which will be $I_{111}, I_{112}, I_{121}, I_{211}, I_{222}, I_{221}, I_{122}, I_{212}$) of length $\frac{1}{3^3} = \frac{1}{27}$ each, and so on. Therefore, the domain of f^n consists of 2^n closed intervals of length $\frac{1}{3^n}$ each.

Hence, from (1), we can say that $\boxed{I_{w_1 w_2 \ldots w_n} = I_{w_1} \cap f^{-1}(I_{w_2}) \cap \ldots \cap f^{-1(n-1)}(I_{w_n})}$, where each w_k is either 1 or 2. Let us verify that for two intervals in the $n = 3$ case:

For the interval I_{111}, $w_1 = 1$, $w_2 = 1$, $w_3 = 1$.
$I_1 \cap f^{-1}(I_1) \cap f^{-2}(I_1) = \left[0, \frac{1}{3}\right] \cap \left[0, \frac{1}{9}\right] \cap \left[0, \frac{1}{27}\right] = \left[0, \frac{1}{27}\right] = I_{111}$.

For the interval I_{112}, $w_1 = 1$, $w_2 = 1$, $w_3 = 2$.
$I_1 \cap f^{-1}(I_1) \cap f^{-2}(I_2) = \left[0, \frac{1}{3}\right] \cap \left[\frac{2}{9}, \frac{1}{3}\right] \cap \left[\frac{2}{27}, \frac{1}{9}\right] = \left[\frac{2}{27}, \frac{1}{9}\right] = I_{112}$.

The others follow similarly.

We notice that for any n, $I_{v_1 v_2 \ldots v_n} \cap I_{w_1 w_2 \ldots w_n} = \phi$ for $(v_1, v_2, \ldots, v_n) \neq (w_1, w_2, \ldots, w_n)$.

5.1 Defining the middle-third Cantor Set

Based on the discussion above, we can say that

$$f^{-1}(D) \cap I_{w_1 w_2 \ldots w_n} = I_{w_1 w_2 \ldots w_n 1} \cup I_{w_1 w_2 \ldots w_n 2}$$

Thus, we can write the domain of definition of the nth iterate of f as

$$D_n = f^{-(n-1)}(D) = \bigcup_{w_1 w_2 \ldots w_n} I_{w_1 w_2 \ldots w_n}$$

Hence, for $n \to \infty$, the domain on which every iterate f^n is defined can be written as

$$C = \bigcap_{n \geq 1} \left(\bigcup_{w_1 w_2 \ldots w_n} I_{w_1 w_2 \ldots w_n} \right) \tag{4}$$

which is the standard middle-third Cantor set.

I couldn't explain why this set is a repeller as claimed in the referred book.

5.2 Dynamics of f

To study some applications of C, let's start with exploring the dynamics of $f : C \to C$. For $x_1, x_2 \in C$ (x_1 and x_2 are very close to one another),

$$d(f(x_1), f(x_2)) = 3d(x_1, x_2)$$

Let us check it:

For $x_1 = 0$ and $x_2 = \frac{1}{9}$, $f(x_1) = 0$ and $f(x_2) = \frac{1}{3}$.

$d(x_1, x_2) = x_2 - x_1 = \frac{1}{9} - 0 = \frac{1}{9}$.

$d(f(x_1), f(x_2)) = f(x_2) - f(x_1) = \frac{1}{3} - 0 = \frac{1}{3}$.

Clearly, $d(f(x_1), f(x_2)) = \frac{1}{3} = 3 \cdot \left(\frac{1}{9}\right) = 3d(x_1, x_2)$.

Therefore, for f^n, we have

$$d(f^n(x_1), f^n(x_2)) = 3^n d(x_1, x_2)$$

for $f^k(x_1)$ and $f^k(x_2)$ in the same interval for $1 \leq k < n$. (Suppose $n = 1$. Let $x_1 = 0$ and $x_2 = \frac{8}{9}$. Then, $f(x_1) = f(0) = 0$ and $f(x_2) = f\left(\frac{8}{9}\right) = \frac{2}{3}$. Then, $d(x_1, x_2) = \frac{8}{9}$, however, $d(f(x_1), f(x_2)) = \frac{2}{3} \neq 3d(x_1, x_2)$.)

Let us verify it:

Fix $n = 2$. For $x_1 = 0$ and $x_2 = \frac{1}{9}$, $f^2(x_1) = f^2(0) = 0$ and $f^2(x_2) = f^2\left(\frac{1}{9}\right) = 1$.

$d(x_1, x_2) = \frac{1}{9}$ (from above).

$d(f^2(x_1), f^2(x_2)) = f^2(x_2) - f^2(x_1) = 1 - 0 = 1$.

Clearly, $d(f^2(x_1), f^2(x_2)) = 1 = 3^2 \cdot \left(\frac{1}{9}\right) = 3^2 d(x_1, x_2)$.

This is an illustration of the butterfly effect, which suggests sensitive dependence on initial conditions - a small change in one state of a system can result in large differences in a later state. It is interesting to note the difference between the divergence of trajectories of the linear map $x \mapsto rx$ and those of the piecewise linear map we're considering presently. The trajectories of the $x \mapsto rx$ map diverge to infinity, while those of the piecewise linear map remain bounded!

A natural question to ask is, 'how big is the Cantor set?' Let us consider its length. From the discussion above, we can say that at the nth iteration, the length of the Cantor set will be $\left(\frac{2}{3}\right)^n$. However, this quantity approaches 0 as $n \to \infty$. If we consider the portion that is removed in every iteration, we see that the total length of the removed portion is

$$\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + ... = \Sigma_{n=1}^{\infty} \frac{2^n}{3^{n+1}} = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1$$

where the first term $a = \frac{1}{3}$ and common ratio $r = \frac{2}{3}$. This means that the probability of choosing a point randomly from the Cantor set is precisely 0.

Thus, we require a different way of 'measuring' the Cantor set. Let us consider the number of points contained in the set and determine whether it is *countable* or *uncountable*. Suppose we want to determine in which interval a point $x \in C$ lies. We can find that by writing

$$x \in I_{w_1} \cap I_{w_1 w_2} \cap ... \cap I_{w_1 w_2 ... w_n} \cap ...$$

This defines a map $\phi : C \to \Sigma_2^+$, where $\Sigma_2^+ = \{1, 2\}^{\mathbb{N}} = \{(w_k)_{k=1}^{\infty} | w_k = 1\text{or } 2,\ \forall\ k \geq 1\}$. As we shall see, this map is a bijection.

5.2.1 Conjugacy 2.0

Let us introduce an equivalence between C and Σ_2^+. Recall

$$I_{w_1 w_2 ...} = I_{w_1} \cap f^{-1}(I_{w_2}) \cap ...$$

Call it $h(w)$. Applying f to this h we get

$$
\begin{aligned}
f(h(w)) &= f(I_{w_1} \cap f^{-1}(I_{w_2}) \cap ... \\
&= [0,1] \cap I_{w_2} \cap f^{-1}(I_{w_3}) \cap f^{-2}(I_{w_4}) \cap ... \\
&= I_{w_2} \cap f^{-1}(I_{w_3}) \cap f^{-2}(I_{w_4}) \cap ...
\end{aligned}
$$

Call it $h(w')$ where $w' = (w_2 w_3 ...)$. Here we have defined a *shift map*

$$\sigma : \Sigma_2^+ \to \Sigma_2^+$$

so that

$$\sigma(w_1 w_2 ...) = (w_2 w_3 ...)$$

We observe that $f(h(w)) = h(\sigma(w))$. That is, either we first apply σ to go from Σ_2^+ to Σ_2^+ and then apply h that will take us from Σ_2^+ to C, or we first apply h to go from Σ_2^+ to C and then apply f that will take us from C to C; we're at the same place both ways. Thus h is a conjugacy between the conjugates f and σ.

5.3 Exercise 1.4 - Binary expansion

Let us define a map $f : [0,1] \to \Sigma_2^+$ where $\Sigma_2^+ = \{w_k | (w_k)_{k=1}^{\infty}$ is 1 or 2 for $k \geq 1\}$, such that $f(x) = \Sigma_{i=0}^{\infty}(x_i \cdot 2^{-i})$ is the binary expansion of real numbers. We know that every real number has a binary expansion. Thus, $\forall x \in [0,1], f(x) \in \Sigma_2^+$. Since $f(x)$ is bijective, $[0,1]$ and Σ_2^+ have the same cardinality and thus, so does the Cantor set.

5.4 Exercise 1.5 - Periodic points

From the definition of the shift map, we have that for $x = 0.a_1 a_2 ...$, $f(x) = 0.a_2 a_3 ....$

We are looking at $\bigcup_k \{x : f^k(x) = x\}$. Using the binary expansion of Cantor sets, we conclude that $f^k(x) = x$ has 2^k solutions.

Let us verify it using a graphical representation.

From figure 1.12 in the referred book, we see that the graph of $f^1(x) = f(x)$ intersects the bisectrix at $2^1 = 2$ points. Hence, we have two periodic points for $f(x)$.

From figure 1.14 in the referred book, we see that the graph of $f^2(x)$ intersects the bisectrix at $2^2 = 4$ points. Hence, we have four periodic points for $f^2(x)$.

Thus, we observe that for a fixed n, $f^n(x)$ intersects the bisectrix at 2^n points, so we have 2^n periodic points.

6 A Topological Consideration

In this section, we explore the cantor set from a topological point of view. (See appendix for relevant definitions.)

6.1 Exercise 1.6

Compact: It is sufficient to prove that the Cantor set is bounded and closed. By definition, the middle-third Cantor set is $C = \bigcup_{n=0}^{\infty} C_n$, where we start with $C_0 = [0, 1]$ and each C_{n+1} is constructed by removing the open middle-third of each sub-interval C_n. The interval $[0, 1]$ bounds the Cantor set; the complement of the Cantor set is the set of open intervals removed in the process. Hence, the Cantor set is closed.

Perfect: Since the Cantor set is closed, we have that $C = \bar{C}$, where $\bar{C}$ denotes the closure of C. Hence, the Cantor set is perfect.

Totally disconnected: Let there be points $x, y \in C$, and let the distance between them be denoted as $|x - y|$. We know that the length of an interval $C_n \in C$ is given by $\frac{1}{3^n}, n = 0, 1, 2, ...$ By the Archimedean property, $\exists$ some n such that $|x - y| < \frac{1}{3^n}$. This suggests that x and y lie in different intervals, and we know that the Cantor set is a union of disjoint closed sets. Hence, it is totally disconnected.

7 Summary

In this report, we developed a basic understanding of fractals and briefly discussed their properties: infinite length and non-differentiability. We saw that the von Koch curve demonstrated these properties. We drew parallels between the construction of the von Koch curve and that of the middle-third Cantor set and discovered that the presence of a fractal is closely related to the chaotic behavior of trajectories in dynamical systems. We learned to determine the trajectory of a point on a given map. Then, we built towards the binary representation of the middle-third Cantor set using a piecewise linear map. And finally, we looked at some primary topological properties of the middle-third Cantor set.

In the following discussion of Cantor sets, we start by examining some more topological properties rigorously and attempt to prove some theorems about *countability*.

8 Appendix

Accumulation point:

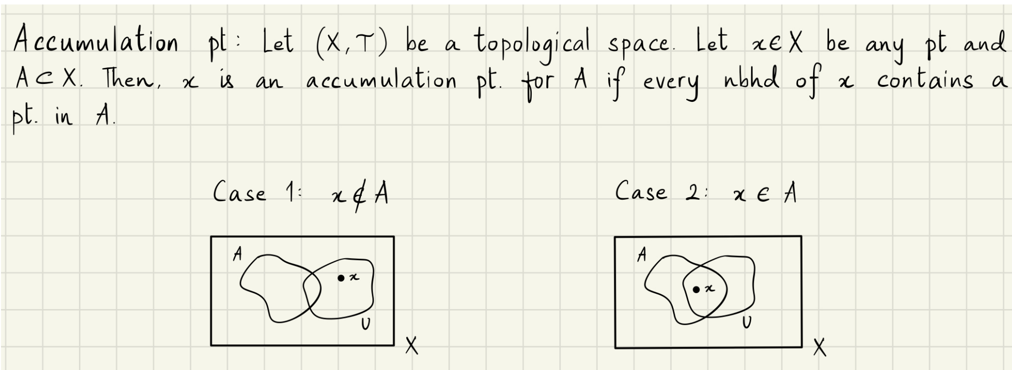

Figure 7: Accumulation Point

Compactness: For our purposes, if a set is closed and bounded, it is compact. (Proof omitted.)

Convergence: Suppose x_n is a sequence in a topological space X. It is said to *converge* to some $x \in X$ if for every neighborhood U of x, $\exists\ N \in \mathbb{N}$ such that $x_n \in U\ \forall\ n \geq \mathbb{N}$. Shown below is an example/illustration:

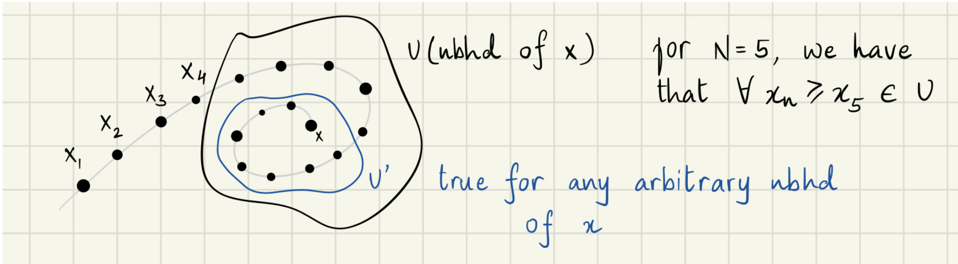

Figure 8: Convergence

Homeomorphism: If $f : X \to Y$ is a map from a topological space X to a topological space Y, then f is a *homeomorphism* if:

- f is bijective
- f is continuous
- f^{-1} exists and is continuous

We show that the above is an equivalence relation:

Reflexive: We want to show that $X \sim X$.

Suppose f is the identity map. Then, $id_x : X \to X$ so that $id_x = x$ is clearly a homeomorphism.

Symmetric: We want to show that if $X \sim Y$, then $Y \sim X$.

If $f : X \to Y$ is a homeomorphism, $f^{-1} : Y \to X$ exists by definition. Thus, $Y \sim X$.

Transitive: We want to show that if $X \sim Y$, $Y \sim Z$, then $X \sim Z$. Let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms. Then, $g(f(x)) : X \to Z$ is also a homeomorphism [2].

Perfect: A set $A \subset X$ where X is a topological space, is said to be *perfect* if every point $x \in A$ is an accumulation point for A.

Procedure to determine the trajectory of x: We start with a point, say x_0, on the x-axis. Then, find $f(x_0)$; i.e., follow the vertical line through the point $(x_0, 0)$ until it intersects the graph of f. This point is $(x_0, f(x_0))$. Now we follow the horizontal line from $(x_0, f(x_0))$ until it reaches the bisectrix. This point is $(f(x_0), f(x_0))$. Repeating the

[2] If f and g are homeomorphisms, they are bijective, continuous, and have continuous inverses and so $g(f(x))$ is also bijective, continuous, and $g(f(x))^{-1} = f^{-1}(g^{-1})$ is also continuous.

process gives us points $(f(x_0), f^2(x_0))$, $(f^2(x_0), f^2(x_0))$ and so on.

Totally disconnectedness: If for any $x, y \in A \subset X$ where X is a topological space, $\exists$ disjoint open sets U, V such that $x \in U$ and $y \in V$ and $A \subset U \cup V$, then A is said to be *totally disconnected.*