l

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Groups of Special Units

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Amanda M. Beeson

Committee in charge:

Professor Harold Stark, Chair
Professor Patrick Diamond
Professor Ron Evans
Professor Russell Impagliazzo
Professor Audrey Terras

2009

The dissertation of Amanda M. Beeson is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2009

DEDICATION

To my family.

EPIGRAPH

Les charmes enchanteux de cette sublime science ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir.

— *C. F. Gauss*

TABLE OF CONTENTS

Signature Page . . . iii

Dedication . . . iv

Epigraph . . . v

Table of Contents . . . vi

Acknowledgements . . . viii

Vita and Publications . . . ix

Abstract of the Dissertation . . . x

Chapter 1 Introduction and Background . . . 1
1.1 Introduction . . . 2
1.2 Dirichlet's analytic class number formula . . . 2
1.3 The Stark conjectures . . . 3
1.4 The related work of Anderson, Kubert, and Sinnott . . . 4
1.5 Results . . . 5
1.5.1 The cyclotomic setting . . . 5
1.5.2 The imaginary quadratic setting . . . 5
1.5.3 Future directions . . . 6

Chapter 2 The Cyclotomic Case . . . 7
2.1 Introduction . . . 8
2.2 Distributions . . . 9
2.2.1 Definitions . . . 9
2.2.2 The universal ordinary distribution . . . 10
2.2.3 Lifting operators . . . 10
2.2.4 The structure of universal distributions . . . 11
2.2.5 Examples . . . 12
2.3 The circular numbers . . . 14
2.4 Index calculations . . . 16
2.4.1 The index of the circular units in the full unit group 16
2.4.2 The Stickelberger ideal . . . 16
2.4.3 A word about Sinnott's proof . . . 17
2.5 The maximal almost abelian extension of $\mathbb{Q}$ . . . 18
2.5.1 Γ-monomials . . . 18
2.5.2 Almost abelian groups . . . 19
2.5.3 The maximal almost abelian extension of $\mathbb{Q}$ . . . 19
2.5.4 A brief explanation of Anderson's proof . . . 20

2.6 First, an example 21
2.7 Results 23
2.7.1 A character combination of Hurwitz zeta functions 24
2.7.2 Trigonometric relations 25
2.7.3 Explicit square roots 26

Chapter 3 The Imaginary Quadratic Case 28
3.1 Introduction 29
3.2 Modular functions and the group of modular units 30
3.2.1 Modular functions 30
3.2.2 The full tower of modular functions $\mathcal{F}$ 31
3.3 The Siegel group 32
3.3.1 The Kronecker limit formulae 33
3.4 The Work of Daniel Kubert 33
3.5 Towards an explicit formula for a square unit 34
3.5.1 The odd relations 34
3.5.2 The distribution relations 35
3.5.3 Explicit squares 36
3.6 On the level of a square root 39

Chapter 4 Conclusion 42
4.1 Concluding remarks 43
4.2 Future work and question 44

Bibliography 46

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Harold Stark, for all of his help, advice, and insight. I would like to thank Professor Cristian Popescu for his help as well. I would like to thank Professor Lance Small for his advice. I would like to thank my committee for their support. I would like to thank Emmy Small for all the help writing up my thesis. And, last but not least, I would like to thank my family for their unwavering belief in me.

VITA

2003	B. S. in Mathematics with Computer Science, Massachusetts Institute of Technology.
2003-2008	Graduate Teaching Assistant, University of California, San Diego.
2008	Visiting Assistant Professor, University of San Diego.
2009	Visiting Assistant Professor, University of California, San Diego.
2009	Ph. D., Mathematics, University of California, San Diego.

ABSTRACT OF THE DISSERTATION

Groups of Special Units

by

Amanda M. Beeson
Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Harold Stark, Chair

Our work extends Anderson's construction of the maximal almost Abelian extension of $\mathbb{Q}$ and Kubert's proof that the Siegel group generates the full unit group up to 2-cotorsion. It is related to Sinnott's index calculations and relies heavily on the machinery of distributions developed by Kubert.

In the cyclotomic setting, we prove the second order vanishing of a character combination of Hurwitz zeta functions and calculate the lead term. From this we derive a family of new trigonometric identities. Finally, we give a general algorithm for finding an explicit square root of a certain combinations of circular numbers that have a square root.

In the imaginary quadratic setting, we give a combination of Siegel units that has a square root. We prove that if the square root of a modular unit has a level then that level is twice the level of the function itself.

Chapter 1

Introduction and Background

1.1 Introduction

Inspired by his observations that every field extension of the rational numbers with abelian Galois group is contained in a pure cyclotomic field and that a single holomorphic function, $f(z) = e^{2\pi i z}$, evaluated at rational numbers is responsible for every such extension, Leopold Kronecker (1823-1891) dreamt that such a function might be found to thus classify abelian extensions of imaginary quadratic fields. His dream came true in the form of complex multiplication, hailed by David Hilbert (1862-1943) as "not only the most beautiful part of mathematics but also of all science" [Hil43]. In fact, the natural generalization, namely to find a function or family of functions whose special values generate the maximal abelian extension of any number field, became Hilbert's twelfth problem.

In [Sta80] Harold Stark put forth a conjectural program about the special values of L-functions that, in some settings, offers a solution to Hilbert's twelfth problem. In that way, the Stark conjectures, and the concomitant study of special values of L-functions and groups of units, have become one of the main approaches to explicit class field theory.

1.2 Dirichlet's analytic class number formula

The mingling of algebra and analysis that Hilbert so revered is typified by the analytic class number formula, which relates the first nonzero Taylor series coefficient of the Dedekind zeta function to algebraic invariants of the number field K. Let K/k be a Galois extension of number fields. The Dedekind zeta function associated to K is defined by its Euler product in the right half-plane $\Re(s) > 1$, where it factors as the product of its constituent Artin L-functions

$$\zeta_K(s) = \prod_p (1 - \mathcal{N}p^{-s})^{-1} = \prod_\rho L_{K/k}(s, \rho)^{\deg(\rho)}.$$

The absolute norm of p is denoted $\mathcal{N}(p)$, the first product is taken over prime ideals p of the ring of integers of K, and the second product is taken over irreducible representations of the Galois group of K/k.

Dirichlet's analytic class number formula then says that the Taylor expansion about $s = 0$ of the Dedekind zeta-function is

$$\zeta_K(s) = -\frac{hR}{W} s^{r_1+r_2-1} + O(s^{r_1+r_2}),$$

where r_1 and r_2 are the number of real and half the number of complex embeddings of K, respectively, h is the class number of K, R the regulator of K, and W the number of roots of unity in K.

1.3 The Stark conjectures

If the Galois group of K/k is abelian, the irreducible representations are all one-dimensional, and

$$\zeta_K(s) = \prod_{\chi} L_{K/k}(s, \chi),$$

where the product is now over irreducible characters of $Gal(K/k)$. Stark wanted to split the leading Taylor coefficient at $s = 0$, namely $-\frac{hR}{W}$, into character components in accordance with the L-function factorization. More specifically, he wanted to find units that naturally split the regulator matrix into character components in correspondence with the L-function decomposition of the zeta function. Furthermore, he wanted to see whether the leading term of each L-function factors into a transcendental part corresponding to the regulator and a rational (or at worst algebraic) part corresponding to $-h/W$ in the class number formula.

To state the Stark conjectures we need to set the stage a bit. Let K/k be an abelian extension of number fields. Let S be a finite set of primes in k containing all infinite and ramified primes of k, and at least one prime, say v, that splits completely from k to K, and at least two primes overall. Let $L_S(s, \chi)$ be the Dirichlet L-function associated to $\chi \in \widehat{G}$ with Euler-factors associated to primes in S removed. Then Stark conjectures that there exists an S-unit $\epsilon \in K$, unique up to roots of unity, such that

$$L'_S(0,\chi) = -\frac{1}{W}\sum_{\sigma} \overline{\chi}(\sigma)\log|\epsilon|_{w^\sigma}, \quad \forall \chi \in \widehat{G}.$$

The sum on the right is over σ in the Galois group of K/k and w is a fixed prime above v. In this setting, Stark further conjectures that the W^{th} root of ϵ generates an abelian extension not only over K, but over k, and experimental data confirms this.

1.4 The related work of Anderson, Kubert, and Sinnott

In [And02], Anderson defines an *almost abelian group* to be one such that every commutator is central and squares to the identity. He defines $G^{ab+\epsilon}$ to be the quotient of $G(\overline{\mathbb{Q}}/\mathbb{Q})$ universal for continuous homomorphisms to almost abelian profinite groups. The corresponding Galois extension of $\mathbb{Q}$ is the compositum of all quadratic extensions of $\mathbb{Q}^{ab}$ that are Galois over $\mathbb{Q}$. By way of what he calls the Main Formula, he shows that this extension, which he calls $\mathbb{Q}^{ab+\epsilon}$, is in fact, $\mathbb{Q}^{ab}$ with the fourth roots of rational primes and certain gamma-monomials adjoined. As he remarks,

> The relations standing between the Main Formula, the index formulas of Sinnott, Deligne reciprocity, the theory of Fröhlich, the theory of Das, the theory of the group cohomology of the universal ordinary distribution and Stark's conjecture and its variants deserve to be thoroughly investigated. We have only scratched the surface here. Stark's conjecture is relevant in view of the well known expansion
>
> $$\sum_{n=0}^{\infty}\frac{1}{(n+x)^s} = \frac{1}{2} - x + s\log\frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$
>
> of the Hurwitz zeta function at $s = 0$" [1].

Anderson also says

> Perhaps there is an analogue of the Main Formula over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing [And02].

Kubert's work on the universal ordinary distribution plays a key role in Anderson's proof and the universal ordinary distribution, although he does not name it that, is at the heart of Sinnott's proof in [Sin78] as well. Kubert proves in [Kub79b] that the universal ordinary distribution is free and in [Kub79a] he calculates its $\mathbb{Z}/2\mathbb{Z}$-cohomology.

1.5 Results

1.5.1 The cyclotomic setting

We link this second order vanishing of L-functions associated to even imprimitive characters to the second order vanishing of a character combination of Hurwitz zeta functions and the consequent trigonometric identities we prove in Chapter Two. Moreover, these identities are a consequence of the explicit expressions for square units that we find. We prove the following results.

1. In Section 2.7.1 we prove the vanishing of a certain class of character sums of Hurwitz zeta functions.

2. In Section 2.7.2 we prove a family of associated trigonometric identities.

3. In Section 2.7.3 we find an explicit expression for the square root of a unit that is guaranteed to be a square.

1.5.2 The imaginary quadratic setting

In Chapter Three we treat the imaginary quadratic setting, where we prove the following results.

1. We prove in Section 3.6 that if the square root of a modular unit has a level then that level is no more than twice the level of the original modular unit.

2. We also give in Section 3.5.3 an explicit expression for square roots of certain combination of Siegel units analogous to the cyclotomic case.

1.5.3 Future directions

Finally, in Chapter Four, we outline the future directions for this work. We aim to construct the maximal almost abelian extension of an imaginary quadratic field. We would also like to find a Galois-module of index equalling exactly the class number in the group of units in the cyclotomic setting.

Chapter 2

The Cyclotomic Case

2.1 Introduction

In this chapter we will introduce the machinery of distributions, use the circular numbers setting to exemplify a theoretical framework, outline the major results of interest, and explain and prove the following new results.

1. In Section 2.7.1 we prove the vanishing of a certain class of character sums of Hurwitz zeta-functions.

2. In Section 2.7.2 we prove a family of associated trigonometric identities.

3. In Section 2.7.3 we find an explicit expression for the square root of a unit that is guaranteed to be a square.

First, though, a few definitions. Throughout, ζ_m denotes a primitive mth root of unity for $m \in \mathbb{N}$. By convention, if m is even we assume $4|m$ because then $\zeta_m = e^{2\pi i/m}$ determines the mth cyclotomic field, $k = \mathbb{Q}(\zeta_m)$, uniquely. Let $k^+ = \mathbb{Q}(\zeta_m)^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ be the maximal real subfield of the mth cyclotomic field, k.

Now, $k/\mathbb{Q}$ is a Galois extension and we will denote by G_m the Galois group $\text{Gal}(k/\mathbb{Q})$. Because k^+ is the fixed field of the subgroup generated by the conjugation autormophism, $k^+/\mathbb{Q}$ is Galois and its Galois group will be denoted G_m^+. Recall, $(\mathbb{Z}/m\mathbb{Z})^\times \cong G_m$ via $t \mapsto \sigma_t$, where $\sigma_t(\zeta_m) = \zeta_m^t$.

Let $U_m = \mathcal{O}_k^\times$ be the group of units in k and, similarly, $U_m^+ = \mathcal{O}_{k^+}^\times$ the group of units in k^+. Suppose k has class number h, k^+ has class number h^+ and let $h^- = h/h^+$.

Define the group of *circular numbers* to be

$$C_m = \langle 1 - \zeta \mid \zeta^m = 1, \zeta \neq 1 \rangle$$

and the group of *circular numbers* in the real subfield to be

$$C_m^+ = C_m \cap k^+.$$

If m is not a prime power then $1-\zeta_m \in U_m$, but if m is a prime power, ζ_m is a p-unit with valuation one at the prime above p. Define the group of *circular units* to be

$$E_m = \langle 1-\zeta \mid \zeta^m = 1, \zeta \neq 1 \rangle \cap U_m$$

and the group of *circular units* at the plus level to be

$$E_m^+ = C_m \cap U_m^+.$$

2.2 Distributions

2.2.1 Definitions

Let $M_k = (\mathbb{Q}/\mathbb{Z})^k$ and denote by $[a]$ the class of $a \in \mathbb{Q}^k$ modulo $\mathbb{Z}^k$. If $\mathcal{A}$ is an additive abelian group contained in a ring then an *ordinary distribution* of rank k and weight w from M_k to $\mathcal{A}$ is a map $f : M_k \to \mathcal{A}$ that satisfies

$$w(N) \sum_{Nb=m} f(b) = f(m)$$

for all $m \in M_k$ and all positive integers N. If w is of the form $w(N) = N^s$ we say it is a distribution of degree s. In te rank one, weight one case this reduces to the familiar formula:

$$\sum_{i=0}^{N-1} f\left(\left[\tfrac{a+i}{N}\right]\right) = f([a]).$$

An *odd* (resp. *even*) *distribution* is one that satisfies $f([a]) = -f([-a])$, resp. $f([a]) = f([-a])$.

In Section 2.2.3 it will be helpful to define a more general distribution mapping out of a more complicated group, but typically M_k will suffice.

Several examples will be given at the end of this section to illustrate the ubiquity of the distribution structure.

2.2.2 The universal ordinary distribution

The *universal ordinary distribution* of rank k is an abelian group U and a map $\delta : M_k \to U$ such that, given a distribution $f : M_k \to \mathcal{A}$, there is a map $f_* : U \to \mathcal{A}$ making the following diagram commutative.

$$\begin{array}{ccc} M_k & \xrightarrow{\delta} & U \\ & \searrow^{f} & \downarrow{\scriptstyle f_*} \\ & & \mathcal{A} \end{array}$$

If $\mathcal{A}$ is in a ring in which 2 is invertible and $\frac{1}{2} \in \mathcal{A}$ then the universal distribution mapping into $\mathcal{A}$ splits up as the direct sum of an odd part and an even part. We call the summands the *universal odd distribution* and the *universal even distribution*, respectively.

2.2.3 Lifting operators

Let p be a prime number and k a natural number. By local class field theory there exists a unique unramified extension K_p of degree k over $\mathbb{Q}_p$. Let $\mathcal{O}_p$ be the ring of integers in K_p and $\mathcal{O}_p^\times$ the units in $\mathcal{O}_p$. We call $C_p = \mathcal{O}_p^\times$ the *Cartan group* at p. Choosing a $\mathbb{Z}_p$-basis for $\mathcal{O}_p$ gives an embedding of $\mathcal{O}_p^\times$ into $GL_n(\mathbb{Z}_p)$.

Let $N = \prod p^{n(p)}$ be a natural number. Define $\mathcal{O}_N = \prod_{p|N} \mathcal{O}_p$ and $\mathbb{Z}_N = \prod_{p|N} \mathbb{Z}_p$. Then, as in the prime case, the composite *Cartan group* is

$$C_N = \mathcal{O}_N^\times = \prod_{p|N} \mathcal{O}_p^\times \hookrightarrow GL_n(\mathbb{Z}_N)$$

Define $\mathcal{O}(N)$ to be the reduction of $\mathcal{O}_N$ modulo N and, analogously, $C(N)$ to be the reduction of C_N modulo N. In other words,

$$C(N) = \mathcal{O}(N)^\times = (\mathcal{O}_N / N\mathcal{O}_N)^\times$$

Define

$$K = \oplus_p K_p$$

and

$$\mathcal{O} = \oplus_p \mathcal{O}_p$$

and then define Z_N to be the elements of $K/\mathcal{O}$ whose order divides N.

Let $T_p : \mathcal{O}_p \to \mathbb{Z}_p$ be a surjective $\mathbb{Z}_p$-linear homomorphisms for each prime p and glue these maps together into $T : \prod_p \mathcal{O}_p \to \prod_p \mathbb{Z}_p$. Let h be an ordinary distribution of weight w on $\mathbb{Q}/\mathbb{Z}$. Then define the *lifted distribution* of h by T to be

$$h \circ T = h\left(\tfrac{T(Nx)}{N}\right).$$

This new distribution, $h \circ T$, is a distribution on $\frac{1}{N}\mathcal{O}/\mathcal{O}$ of degree $w(N)N^{1-k}$. If h is degree $k-1$ then $h \circ T$ is degree 0.

2.2.4 The structure of universal distributions

There are several different important constructions that fall under this name. By abuse of notation we will identify only the target abelian group with the distribution itself.

First, let R be the set of all distribution relations on elements of M_k for all N. Then M_k/R is the universal ordinary distribution. Further, if one adds to R the odd relations for all $[a] \in M_k$

$$[a] + [-a]$$

then the resulting object is the universal odd distribution. Likewise, adding to R the even relations

$$[a] - [-a]$$

for all $[a] \in M_k$ makes M_k/R the universal even distribution.

As mentioned above, sometimes it is of benefit to work with distributions on a more complicated group, namely $K/\mathcal{O}$. In this setting, we introduce the Stickelberger distribution associated to the lifted (in the sense of Section 2.2.3) k^{th} Bernoulli polynomial and it turns out to be the universal even or odd distribution, as k is even or odd.

Given a distribution $h : Z_N \to \mathcal{A}$ we define the Stickelberger distribution associated to h to be

$$\mathrm{St}_h(x) = \sum_{a \in C(N)} h(ax)\sigma_a^{-1}.$$

Note that St_h takes values in the group ring $\mathcal{A}[G(N)]$, where $C(N)$, the Cartan group, is isomorphic to the group $G(N)$ via $a \mapsto \sigma_a$.

2.2.5 Examples

The following are just a few of the familiar objects that are, in fact, distributions. Let $\langle x \rangle$ be the unique real number x', $0 \leq x' < 1$ such that $x - x'$ is an integer.

First Bernoulli polynomial

The map coming from the first Bernoulli polynomial $B_1 : x \mapsto \langle x \rangle - \frac{1}{2}$ is easily seen to be a distribution of rank one and weight one.

Second Bernoulli polynomial

The map coming from the second Bernoulli polynomial

$$B_2 : x \mapsto \langle x \rangle^2 - \langle x \rangle + \frac{1}{6}$$

is a rank one, degree one distribution on $\mathbb{Q}/\mathbb{Z}$ because

$$N \sum_{i=0}^{N-1} B_2\left(\tfrac{a+i}{N}\right) = B_2(a).$$

Bernoulli distributions in general

More generally, the k^{th} Bernoulli polynomial defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=1}^{\infty} B_k(x)\tfrac{t^k}{k!}$$

gives a degree $k-1$ distribution on $\mathbb{Q}/\mathbb{Z}$ and, by the lifting process described in Section 2.2.3, one can construct a related degree 0 distribution.

The circular numbers

The map $e(x) = 1 - e^{2\pi i x}$ is a distribution under the operation of multiplication because the polynomial $1 - x^N$ factors as

$$1 - x^N = \prod_{r=0}^{N-1} (1 - e^{\frac{2\pi i r}{N}} x) = \prod_{r=0}^{N-1} (1 - \zeta_N^r x)$$

so it follows by setting $x = \frac{a}{N}$ that

$$\prod_{r=0}^{N-1} e\left(\frac{a+r}{N}\right) = \prod_{r=0}^{N-1} (1 - e^{\frac{2\pi i (a+r)}{N}}) = \prod_{r=0}^{N-1} (1 - e^{\frac{2\pi i r}{N}} e^{\frac{2\pi i a}{N}}) = 1 - (e^{\frac{2\pi i a}{N}})^N = e(a)$$

The Dirichlet map

It is an immediate corollary to the previous example that the map $\lambda : \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ given by

$$\lambda : r \mapsto -\frac{1}{2} \log |1 - e^{2\pi i r}|$$

is a distribution. Taking the logarithm, though, makes it look more familiar because the target Abelian group is now additive so

$$\sum_{r=0}^{N-1} \lambda(\tfrac{a+r}{N}) = \sum_{r=0}^{N-1} -\frac{1}{2} \log |e\left(\tfrac{a+r}{N}\right)| = -\frac{1}{2} \log |\prod_{n=0}^{N} e\left(\tfrac{a+r}{N}\right)|,$$

which is none other than

$$-\frac{1}{2} \log |e(a)| = \lambda(a).$$

The Siegel units

The function $\Phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathcal{F}_N$ taking (u, v) to the Siegel function $\phi(u, v)$ modulo constants, a function we will study in the following chapter, is a rank two, weight one distribution.

The Stickelberger distribution

The Stickelberger distribution associated to given distribution h is another important number theoretic distribution that turns out to be the universal ordinary odd or even (for odd or even k, resp.) distribution for $h = B_k$.

The Hurwitz zeta function

The Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

is defined for real x with $0 < x \leq 1$. So defining $\{x\}$ to be the unique representative x' of x mod $\mathbb{Z}$ such that $0 < x' \leq 1$, we see that

$$x \mapsto \zeta(s, \{x\})$$

is an ordinary distribution of degree $-s$ on $\mathbb{Q}/\mathbb{Z}$. In other words,

$$N^{-s} \sum_{i=0}^{N-1} \zeta(s, \left\{\tfrac{a+i}{N}\right\}) = \zeta(s, \{a\}).$$

2.3 The circular numbers

We outline here some key properties of the circular numbers.

Generators and Relations

In [Bas66], Bass gives a complete set of generators and relations for the circular numbers modulo torsion:

$$C_m/\sim = \langle 1-\zeta | \zeta^m = 1, \zeta \neq 1\rangle/\sim$$

where $a \sim b \iff a = \mu b$ for μ a root of unity.

Let $e : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times/\sim$ be the map

$$e(x) = 1 - e^{2\pi i x}.$$

In [Bas66] Bass proves that the only relations among the circular numbers (modulo roots of unity) are

1. The distribution relations (written multiplicatively, in this case):

$$\prod_{r=0}^{N-1} e\left(\tfrac{a+r}{N}\right) = e(a)$$

   for any natural number N and for all $a \in \mathbb{Q}/\mathbb{Z}$.

2. And the even relations (modulo torsion, namely, the root of unity ζ),

$$e(-a) = \zeta e(a)$$

   for every $a \in \mathbb{Q}/\mathbb{Z}$.

Universal even distribution

This proves that $C_m/\sim$ is the universal even distribution up to 2-torsion, which is not accounted for because in the universal even distribution, for example, $[\frac{1}{4}] = [\frac{3}{4}]$ so

$$2[\tfrac{1}{4}] = [\tfrac{1}{4}] + [\tfrac{3}{4}] = [1] = 0,$$

but in C_m, we have

$$e(1/4)^2 \sim 2,$$

which is not 1 modulo roots of unity, hence not torsion. However, if we consider, $\mathcal{A}'$, the group of elements of $\mathbb{Q}/\mathbb{Z}$ generated by elements $[a] \in \mathbb{Q} \setminus \frac{1}{2}\mathbb{Z}$ we see that $C_m/\sim$ is $\mathcal{A}'$ modulo the distribution relations and the even relations.

G_N-module

Recall, $k = \mathbb{Q}(\zeta_N)$ and G_N is the Galois group $\mathrm{Gal}(k/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$. For any $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ let $\sigma_t \in \mathrm{Gal}(k/\mathbb{Q})$ be the corresponding automorphism. Then for any $a \in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$

$$e(a)^{\sigma_t} = e(ta)$$

so the map e commutes with the action of G_N.

Index in units

In Section 2.4.1 below, we discover that the index of the real circular units in U_m^+ is h^+.

Square roots

Finally, as we shall see in Section 2.5, the square roots of certain circular units generate the maximal almost abelian extension of $\mathbb{Q}$.

The entire framework outlined above will be recapitulated in the imaginary quadratic setting in Section 3.6, where it will become clear that the two cases are not merely analogous, but that they rely subtly on one another.

2.4 Index calculations

2.4.1 The index of the circular units in the full unit group

Sinnott showed in [Sin78] that the circular units, E_m^+ have finite index in the full group of units, U_m^+, in k^+. More specifically,

$$[U_m^+ : E_m^+] = 2^a h^+$$

where a is defined as follows. Let g be the number of distinct primes dividing m. Then $a = 0$ if $g = 1$ and $a = 2^{g-2} - 1$ if $g > 1$.

Note that if m is a prime power or the product of two primes powers the circular units have index in U_m^+ equal to the class number of k^+.

2.4.2 The Stickelberger ideal

Recall, we let G_m be the Galois group of $k = \mathbb{Q}(\zeta_m)$ over $\mathbb{Q}$. Then $(\mathbb{Z}/m\mathbb{Z})^\times \cong \mathrm{Gal}(k/\mathbb{Q})$ via the map $t \mapsto \sigma_t$ where $\sigma_t(\zeta_m) = \zeta_m^t$.

The *Stickelberger ideal* is an ideal of the group ring $R = \mathbb{Z}[G_m]$ defined by Sinnott in [Sin78] as follows. Recall we set $\langle x \rangle$ to be the unique representative x' of x mod $\mathbb{Z}$ with $0 \leq x' < 1$. Define the Stickelberger element:

$$\theta(a) = \sum_{\substack{t \mod m \\ (t,m)=1}} \langle -\frac{at}{m} \rangle \sigma_t^{-1} \in \mathbb{Q}[G_m].$$

Let S' be the subgroup of $\mathbb{Q}[G_m]$ generated by $\theta(a)$ for all $a \in \mathbb{Z}$. Since $\sigma_t(\theta(a)) = \theta(ta) \in \mathbb{Z}[G_m]$ we see S' is closed under the action of $\mathbb{Z}[G_m]$, hence *the Stickelberger ideal of* k, which is defined to be

$$S = S' \cap \mathbb{Z}[G_m]$$

is actually an ideal of $\mathbb{Z}[G_m]$.

Defining $\mathbb{Z}[G_m]^-$ and S^- to be the submodules of $\mathbb{Z}[G]$ and S for which $j(x) = -x$, where $j \in G_m$ is complex conjugation. Sinnott proves in [Sin78] that the minus part of the Stickelberger ideal has index h^- in $\mathbb{Z}[G_m]^-$.

We note here that dotting with the idempotent $e^- = \frac{1+j}{2}$ gives

$$\theta(a) \cdot e^- = \sum_{\substack{t \bmod m \\ (t,m)=1}} \left(-\langle \tfrac{at}{m} \rangle + \frac{1}{2} \right) \sigma_t^{-1},$$

which is the Stickelberger distribution associated to the (lifted) first Bernoulli polynomial.

2.4.3 A word about Sinnott's proof

In Sinnott's proofs of the index of the real circular units in the full unit group U_m^+ and the index of the Stickelberger ideal in the group ring, he uses successive index calculations of submodules of the group ring. One of these submodules is the universal odd distribution and his work is the first place the universal odd distribution appears in the literature, even though he does not refer to it as such. It would be interesting to reinterpret his work in terms of distributions and perhaps compare it with the work of Kubert on the structure of universal distributions.

2.5 The maximal almost abelian extension of $\mathbb{Q}$

2.5.1 Γ-monomials

Let $\mathcal{A}$ be the free group on symbols of the form $[a]$, where $[a]$ is the class of $a \in \mathbb{Q}$ modulo the relation $[a] = [b] \iff b - a \in \mathbb{Z}$. Define $\mathbf{\Gamma} : \mathcal{A} \to \mathbb{R}^\times$ to be the unique homomorphism such that

$$\mathbf{\Gamma}([a]) = \begin{cases} \frac{\sqrt{2\pi}}{\Gamma(a)} & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0. \end{cases}$$

Letting

$$Y_p[a] = \sum_{i=0}^{p-1} \left[\frac{a+i}{p}\right] - [a]$$

the algebraic Γ-relations can be rewritten in terms of the homomorphisms $\mathbf{\Gamma}$, and the operator Y_p as follows.

$$\mathbf{\Gamma}([a] + [-a]) = 2\sin(\pi a)$$
$$\mathbf{\Gamma}(Y_p[a]) = p^{1/2-a}$$

for $a \in \mathbb{Q} \cap (0, 1)$.

Now define the symbol $\mathbf{a}_{pq}$ for $2 < p < q$ to be

$$\mathbf{a}_{pq} = \sum_{i=1}^{\frac{p-1}{2}} \left(\left[\frac{i}{p}\right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i}{pq} + \frac{k}{q}\right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q}\right] - \sum_{l=0}^{\frac{p-1}{2}} \left[\frac{j}{pq} + \frac{l}{p}\right] \right)$$

and for $p = 2$ to be

$$\mathbf{a}_{pq} = \left(\left[\frac{1}{4}\right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{1}{4q} + \frac{k}{q}\right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q}\right] + \left[-\frac{1}{2q} + \frac{j}{q}\right] - \left[\frac{j}{2q}\right] - \left[-\frac{1}{4q} + \frac{j}{2q}\right] \right)$$

Then define the map

$$\sin : \mathcal{A} \to (\mathbb{Q}^{ab})^\times$$

to be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2\sin(\pi a) = |1 - e^{2\pi i a}| & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0. \end{cases}$$

For $p < q$, it follows from work of Das in [Das00] that

$$\frac{\mathbf{\Gamma}(\mathbf{a}_{pq})}{\sqrt{\sin \mathbf{a}_{pq}}} = \begin{cases} p^{\frac{-(q-1)^2}{16q}} q^{\frac{(p-1)^2}{16p}} & \text{if } 2 < p \\ 2^{-\frac{q-1}{8}} q^{\frac{1}{8}} & \text{if } p = 2. \end{cases}$$

Thus, certain $\mathbf{\Gamma}$-monomials are related to monomials in the square root of the sine function, and one can begin to see that the leading terms of L-series, in which Γ-values shows up, are related to extensions of cyclotomic fields.

2.5.2 Almost abelian groups

A group G is called *almost abelian* if every commutator ($[x, y] = xyx^{-1}y^{-1}$) is central and squares to the identity. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ and $\mathbb{Q}^{ab}$ be the maximal abelian extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. We define G^{ab} be the Galois group of $\mathbb{Q}^{ab}$ over $\mathbb{Q}$ and $G^{ab+\epsilon}$ to be the quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that is universal for homomorphisms to almost abelian profinite groups. If $\mathbb{Q}^{ab+\epsilon}$ is the corresponding Galois extension of $\mathbb{Q}$ then it turns out that $\mathbb{Q}^{ab+\epsilon}$ is the compositum of all quadratic extensions of $\mathbb{Q}^{ab}$ that are Galois over $\mathbb{Q}$.

2.5.3 The maximal almost abelian extension of $\mathbb{Q}$

In [And02], Anderson proves that the set

$$\{\sqrt{p}\}_{p\ prime} \cup \{\sin \mathbf{a}_{pq}\}$$

projects onto a $\mathbb{Z}/2\mathbb{Z}$-basis for $H^0(G^{ab}, (\mathbb{Q}^{ab})^\times/(\mathbb{Q}^{ab})^{\times 2})$. He constructs $\mathbb{Q}^{ab+\epsilon}$ explicitly as

$$\mathbb{Q}^{ab+\epsilon} = \mathbb{Q}(\mu \cup \{\sqrt[4]{p}\}_{p\ prime} \cup \{\sqrt{\sin \mathbf{a}_{pq}}\}).$$

where μ is the group of all roots of unity. Note that this construction is of interest because it shows that $\mathbb{Q}^{ab+\epsilon}$ is obtained from $\mathbb{Q}^{ab}$ by adjoining the fourth roots of rational primes and, thanks to the Das relation above, gamma-monomials of the form $\mathbf{\Gamma}(\mathbf{a}_{pq})$.

2.5.4 A brief explanation of Anderson's proof

Anderson's proof in [And02] uses in a critical way the structure of the universal odd distribution U^-, which in this case is $\mathcal{A}$ modulo the relations

$$[a] - \sum_{i=0}^{N-1} \left[\frac{a+i}{N}\right] \text{ and } [a] + [-a]$$

for all $a \in \mathbb{Q}$ and $N \in \mathbb{N}$. In [Kub79b] and [Kub79a], Kubert shows that U^- is torsion-free except for 2-torsion. Kubert shows in [Kub79b] that the universal ordinary distribution is free abelian so the torsion subgroup of U^- is isomorphic to $H^2(\mathbb{Z}/2\mathbb{Z}, U^-)$. In an earlier paper, [And99], Anderson exhibits a $\mathbb{Z}/2\mathbb{Z}$-basis for $H^2(\mathbb{Z}/2\mathbb{Z}, U^-)$, and hence for the torsion subgroup of U^-, which was subsequently refined to be $\{\mathbf{a}_{pq}\}$ for primes $p < q$. Work of Das in [Das00] showed for any torsion element $\mathbf{a} \in U^-$, both $\mathbf{\Gamma}(\mathbf{a})$ and $(\sin \mathbf{a})$ are algebraic.

The cohomology group $H^2(G, E)$ is in bijection with isomorphism classes of extensions of G by E. Anderson defines $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$ to be the quotient of $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})$ by the subgroup of classes of extensions of G^{ab} by $\mathbb{Z}/2\mathbb{Z}$ the middle group of which is abelian. Thus, $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$ is in bijection with Galois groups of almost abelian extensions of $\mathbb{Q}$. He produces a canonical class in $H^2(G^{ab}, G^{\epsilon})$, which he calls

$$\Sigma_G^{\epsilon}:\ 1 \longrightarrow G^{\epsilon} \longrightarrow G^{ab+\epsilon} \longrightarrow G^{ab} \longrightarrow 1\ .$$

Denote by G^{ϵ} the kernel of the natural map from $G^{ab+\epsilon}$ to G^{ab}. Kummer theory identifies $H^0(G^{ab}, (\mathbb{Q}^{ab})^{\times}/(\mathbb{Q}^{ab})^{\times 2})$ with the Pontryagin dual of G^{ϵ}.

Elements $c \in H^1(G^{\epsilon}, \mathbb{Z}/2\mathbb{Z})$ can be viewed as continuous homomorphisms $c : G^{\epsilon} \to \mathbb{Z}/2\mathbb{Z}$. As such, they induce maps $H^2(G^{ab}, G^{\epsilon}) \to H^2(G^{\epsilon}, \mathbb{Z}/2\mathbb{Z})$ given by composition with c $(a \mapsto c \circ a)$.

He proves the composition of maps

$$H^1(G^{\epsilon}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(G^{ab}, G^{\epsilon}) \longrightarrow H^2(G^{\epsilon}, \mathbb{Z}/2\mathbb{Z})$$

is injective; then he proves that the map induced by the cup product is an isomorphism and inverts it explicitly.

Theorem 2.5.1 (Anderson, 2002)**.** *The cup product induces an isomorphism*

$$\wedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$$

of vector spaces over $\mathbb{Z}/2\mathbb{Z}$*, where* $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$ *is defined as above.*

Finally, he shows that

$$H^0(G^{ab}, (\mathbb{Q}^{ab})^\times/(\mathbb{Q}^{ab})^{\times 2}) \longrightarrow \wedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$$

is a sequence of isomorphisms and that the map

$$H^2(\mathbb{Z}/2\mathbb{Z}, U^-) \to H^0(G^{ab}, (\mathbb{Q}^{ab})^\times/(\mathbb{Q}^{ab})^{\times 2})$$

is induced by the homomorphism sin defined above. He then proves that tracing backwards through the sequence of maps laid out above, $\mathbf{a}_{pq}$ maps to $\sin(\mathbf{a}_{pq})$, which is turn maps to a basis vector of $\wedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$, what he calls $e_p \wedge e_q$. Together, the numbers of the form $\sqrt{p}$ for p prime and $\sin(\mathbf{a}_{pq})$ project onto a basis for $H^0(G^{ab}, (\mathbb{Q}^{ab})^\times/(\mathbb{Q}^{ab})^{\times 2})$, which is in bijection with almost abelian extensions of $\mathbb{Q}$, and so the result follows.

2.6 First, an example

Anderson's result in [And02] includes the construction of gamma-monomials that, together with roots of unity, generate almost abelian extensions of $\mathbb{Q}$. Let k be the cyclotomic field $\mathbb{Q}(e^{2\pi i/pq})$ and let $\alpha \in k$ be such that $K = k(\sqrt{\alpha})$ is Galois over $\mathbb{Q}$ (then $K/\mathbb{Q}$ is an almost abelian extension of fields). Let σ be an element of the Galois group of $k/\mathbb{Q}$. Then $\frac{\alpha^\sigma}{\alpha}$ is the square of a unit in k and, thus, has a square root in k. I can find this square root explicitly. For example, take

$$\alpha = \sin(\mathbf{a}_{3\cdot 5}) = \frac{4\sin(5\pi/15)\sin(2\pi/15)}{4\sin(4\pi/15)\sin(3\pi/15)}$$

and $\sigma : \zeta_{15} \mapsto \zeta_{15}^2$ so

$$\alpha^\sigma = \frac{4\sin(10\pi/15)\sin(4\pi/15)}{4\sin(8\pi/15)\sin(6\pi/15)}.$$

Then

$$\frac{\alpha^\sigma}{\alpha} = \frac{\frac{4\sin(10\pi/15)\sin(4\pi/15)}{4\sin(8\pi/15)\sin(6\pi/15)}}{\frac{4\sin(5\pi/15)\sin(2\pi/15)}{4\sin(4\pi/15)\sin(3\pi/15)}} = \left(\frac{2\sin(4\pi/15)}{4\sin(8\pi/15)\sin(2\pi/15)}\right)^2$$

From this, follow a number of corollaries. First, by dividing both sides of the above expression by the square above, I found a family of new trigonometric identities indexed by products of distinct primes pq, exemplified by $pq = 15$:

$$\frac{4\sin(2\pi/15)\sin(3\pi/15)\sin(8\pi/15)}{\sin(6\pi/15)} = 1.$$

Where, if we act by multiplication by 8, we see

$$\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)} = 1.$$

Second, in light of the expansion of the Hurwitz zeta function at $s = 0$,

$$\zeta(s,x) = \frac{1}{2} - x + s\log\frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$

and the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

these identities show that certain sums of Hurwitz zeta functions vanish to second order at $s = 0$. For example,

$$\zeta(s,\tfrac{3}{15}) - \zeta(s,\tfrac{1}{15}) - \zeta(s,\tfrac{4}{15}) - \zeta(s,\tfrac{9}{15}) + \zeta(s,\tfrac{12}{15}) - \zeta(s,\tfrac{14}{15}) - \zeta(s,\tfrac{11}{15}) - \zeta(s,\tfrac{6}{15}) =$$
$$s\log\left(\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)}\right) + O(s^2)$$

vanishes to second order at $s = 0$. Therefore, the coefficient of s^2 is of interest and we can actually calculate that it is $-\log(3)[\frac{1}{2}\log(5) + \log|(1-\zeta_5)(1-\zeta_5^2)^{-1}|]$!

The vanishing of these Hurwitz zeta functions is connected to the known first-order vanishing of L-functions associated to even Dirichlet characters of conductor dividing p. Let χ_p be an even character of conductor dividing p (potentially trivial), and χ_{pq} its inflation to the group $(\mathbb{Z}/pq\mathbb{Z})^\times$. Thus, $\chi_{pq}(a) = \chi_p(a)$ except when q divides a, in which case $\chi_{pq}(a) = 0$. Suppose $S = \{p, q\}$, and recall that $L_S(s, \chi)$ has Euler factors associated to primes in S removed. Then the second order vanishing of $(pq)^s L(s, \chi_p)(1 - q^{-s})$ at $s = 0$ can be expressed in terms of the second order vanishing of Hurwitz zeta functions as in the following example for $S = \{3, 5\}$:

$$L(s, \chi_5)(1 - 3^{-s}) =$$

$$L(s, \chi_5)(1 - \chi_5(3)3^{-s}) + L(s, \chi_5)(\chi_5(3)3^{-s} - 3^{-s}) =$$

$$L_S(s, \chi_{15}) - 3^{-s}L(s, \chi_5) + 3^{-s}\chi_5(3)L(s, \chi_5) =$$

$$15^{-s}\sum_{a=1}^{14}\chi_{15}(a)\zeta(s, \tfrac{a}{15}) - 15^{-s}\sum_{b=1}^{4}\chi_5(b)\zeta(s, \tfrac{3b}{15}) + 15^{-s}\sum_{c=1}^{4}\chi_5(3c)\zeta(s, \tfrac{3c}{15}).$$

Let $\zeta(s)$ be the Riemann zeta function. Then the above combination plus $\zeta(s)(1 - 3^{-s})(1 - 5^{-s})$ is still a function that vanishes to second order, but we have now isolated the $\chi = 1$ terms in the above sum. That is, we have isolated the $a = 1, 4, 11, 14$, $b = 2, 3$, and $c = 1, 4$ terms. We can now see that first non-vanishing coefficient from the previous section is, in fact, the lead term of $\frac{1}{2}(1 - 3^{-s})[\zeta(s)(1 - 5^{-s}) + L(s, \chi_5)]$. This lead term is known (because the Stark conjectures are proved in this setting) to be

$$-\log(3)(\tfrac{1}{2}\log(5) + L'(0, \chi_5)) = -\log(3)[\tfrac{1}{2}\log(5) + \log|(1-\zeta_5)(1-\zeta_5^{-1})| + \log|(1-\zeta_5^2)(1-\zeta_5^{-2})|] =$$

$$-\log(3)[\tfrac{1}{2}\log(5) + \log|(1-\zeta_5)(1-\zeta_5^2)^{-1}|],$$

where ζ_5 is a primitive fifth root of unity.

2.7 Results

Let p and q be distinct primes and set $N = pq$. Suppose χ_p is a Dirichlet character with cond(χ) dividing p. We refer to the imprimitive character modulo N as χ_N. That is, $\chi_N(a) = \chi_p(a)$ for all a relatively prime to N and $\chi_N(a) = 0$

otherwise. Given a set $S = \{p, q\}$ and a Dirichlet character χ, define the incomplete L-function associated to this data, $L_S(s, \chi)$, to be the ordinary L-function associated to χ with the Euler factors for primes in S removed.

2.7.1 A character combination of Hurwitz zeta functions

Theorem 2.7.1 (Beeson, 2009). *Assume N, p, q, χ_p, and χ_N are as above, and that χ_p is an even character. The following character combination of Hurwitz zeta functions vanishes to second order at $s = 0$:*

$$\sum_{a \bmod N} \chi_N(a)\zeta(s, \{\tfrac{a}{N}\}) - \sum_{a \bmod p} \chi_p(a)\zeta(s, \{\tfrac{aq}{N}\})$$

$$+ \sum_{a \bmod p} \chi_p(aq)\zeta(s, \{\tfrac{aq}{N}\}).$$

(Recall, the Hurwitz zeta function was defined in 2.2.5.)

Proof. It is known that $L(s, \chi_p)$ vanishes to first order at $s = 0$ for χ_p an even non-trivial character so $L(s, \chi_p)(1 - q^{-s})$ vanishes to second order. Let $S = \{p, q\}$. We calculate that

$$L(s, \chi_p)(1 - q^{-s}) =$$

$$L(s, \chi_p)(1 - \chi_p(q)q^{-s}) + L(s, \chi_p)(\chi_p(q)q^{-s} - q^{-s}) =$$

$$L_S(s, \chi_N) - q^{-s}L(s, \chi_p) + q^{-s}\chi_p(q)L(s, \chi_p) =$$

$$N^{-s}\sum_{a=1}^{N-1} \chi_N(a)\zeta(s, \tfrac{a}{N}) - N^{-s}\sum_{b=1}^{p-1} \chi_p(b)\zeta(s, \tfrac{qb}{N}) + N^{-s}\sum_{c=1}^{p-1} \chi_p(qc)\zeta(s, \tfrac{qc}{N}) =$$

$$N^{-s}\left[\sum_{a=1}^{N-1} \chi_N(a)\zeta(s, \tfrac{a}{N}) - \sum_{b=1}^{p-1} \chi_p(b)\zeta(s, \tfrac{qb}{N}) + \sum_{c=1}^{p-1} \chi_p(qc)\zeta(s, \tfrac{qc}{N})\right].$$

Thus, the latter combination of Hurwitz zeta functions also vanishes to second order at $s = 0$.

□

2.7.2 Trigonometric relations

Corollary 2.7.2 (Beeson, 2009)**.** *Assume N, p, q, χ_p, and χ_N are as in Theorem 2.7.1. Assume further that χ_p is an even character. Then*

$$\frac{\displaystyle\prod_{\substack{b \bmod p\\ \chi_p(b)=1}} 2\sin(\{\tfrac{bq\pi}{N}\})}{\displaystyle\prod_{\substack{a \bmod N\\ \chi_N(a)=1}} 2\sin(\{\tfrac{a\pi}{N}\}) \prod_{\substack{c \bmod p\\ \chi_p(cq)=1}} 2\sin(\{\tfrac{cq\pi}{N}\})} = 1,$$

where each product is over a set of representatives modulo ± 1*. That is, if* $\sin(x)$ *occurs in one of the products then* $\sin(-x)$ *does not occur in that product.*

Proof. By Theorem 2.7.1,

$$\sum_{a \bmod N} \chi_N(a)\zeta(s, \{\tfrac{a}{N}\}) - \sum_{a \bmod p} \chi_p(a)\zeta(s, \{\tfrac{aq}{N}\})$$
$$+ \sum_{a \bmod p} \chi_p(aq)\zeta(s, \{\tfrac{aq}{N}\})$$

vanishes to order two; hence, the coefficient of s is zero. However, by the expansion at $s = 0$ of the Hurwitz zeta function,

$$\zeta(s, x) = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2),$$

we see that the coefficient of s is

$$\sum_{a \bmod N} \chi_N(a) \log \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} - \sum_{b \bmod p} \chi_p(b) \log \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}} + \sum_{c \bmod p} \chi_p(cq) \log \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}}.$$

By the orthogonality of the characters, we can isolate the terms where $\chi_N(a) = \chi_p(b) = \chi_p(cq) = 1$, to get

$$\sum_{\chi_N(a)=1} \log \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} - \sum_{\chi_p(b)=1} \log \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}} + \sum_{\chi_{p(cq)=1}} \log \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}}$$

$$= \log \frac{\displaystyle\prod_{\chi_N(a)=1} \frac{\Gamma(\{\frac{a}{N}\})}{\sqrt{2\pi}} \prod_{\chi_p(cq)=1} \frac{\Gamma(\{\frac{cq}{N}\})}{\sqrt{2\pi}}}{\displaystyle\prod_{\chi_p(b)=1} \frac{\Gamma(\{\frac{bq}{N}\})}{\sqrt{2\pi}}} = 0.$$

Because χ_p is even, $\chi_p(x) = 1$ if and only if $\chi_p(-x) = 1$; likewise, $\chi_N(x) = \chi_N(-x)$. So the above relation together with the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

give

$$\log \frac{\prod_{\chi_p(b)=1} 2\sin(\{\frac{bq\pi}{N}\})}{\prod_{\chi_N(a)=1} 2\sin(\{\frac{a\pi}{N}\}) \prod_{\chi_p(cq)=1} 2\sin(\{\frac{cq\pi}{N}\})} = 0,$$

where the products are now limited by the pairing of the x and $-x$ terms. In other words, if $\sin(x)$ occurs in one of the products then $\sin(-x)$ does not occur in the same product. This concludes the proof. $\square$

2.7.3 Explicit square roots

Recall first that we defined $\mathcal{A}$ to be the free abelian group on symbols of the form $[a]$ for $[a] \sim [b]$ if and only if $a - b \in \mathbb{Z}$ and

$$\sin : \mathcal{A} \to (\mathbb{Q}^{ab})^\times$$

to be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2\sin(\pi a) = |1 - e(a)| & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0. \end{cases}$$

Let $\sigma \in G^{ab}$ act on $\mathcal{A}$ via

$$\sigma([a]) = [b] \Longleftrightarrow e(a) = e(b).$$

Anderson's result proves that every quadratic extension over $\mathbb{Q}(\zeta_{pq})$ that is Galois over $\mathbb{Q}$ is generated either by $\sqrt[4]{p}$ or $\sqrt[4]{q}$, or by $\sqrt{\sin \mathbf{a_{pq}}}$ or one of its conjugates (where $\mathbf{a}_{pq}$ was defined in 2.5.1). Thus, for every $\sigma \in G_N$ we will have a square circular unit of the form $\frac{\alpha^\sigma}{\alpha}$ for $\alpha = \sin \mathbf{a_{pq}}$. Let $\mathcal{A}'$ be the subgroup of $\mathcal{A}$ generated by symbols $[a]$ for $a \notin \frac{1}{2}\mathbb{Z}$. Anderson refers to the Das cocycle, $\mathbf{c}_\sigma$, which is an element of $\mathcal{A}'$, such that

$$\frac{\sin \mathbf{a}_{pq}^\sigma}{\sin \mathbf{a}_{pq}} = \sin^2 \mathbf{c}_\sigma.$$

We give here an algorithm that gives an explicit expression for the Das cocycle and an interpretation by means of zeta-functions.

First, define $\hat{\zeta}(s,\{x\}) = \zeta(s,\{x\}) + \zeta(s,\{1-x\})$ and

$$\zeta(s,\mathbf{a}_{pq}) = \sum_{i=1}^{(p-1)/2}\left[\hat{\zeta}(s,\{\tfrac{i}{p}\}) - q^{-s}\sum_{k=0}^{(q-1)/2}\hat{\zeta}(s,\{\tfrac{i+kp}{N}\})\right] - \sum_{j=1}^{(q-1)/2}\left[\hat{\zeta}(s,\{\tfrac{j}{q}\}) - p^{-s}\sum_{l=0}^{(p-1)/2}\hat{\zeta}(s,\{\tfrac{j+ql}{N}\})\right]$$

so that $\zeta(s,\mathbf{a}_{pq})$ has $\log(\sin(\mathbf{a}_{pq}))$ as the coefficient of s in its expansion at $s=0$.

Let N, p, q be as in Theorem 2.7.1. For each $\sigma \in G_N = \mathrm{Gal}(\mathbb{Q}(\zeta_N)^+/\mathbb{Q})$, the following algorithm produces the coefficient of s at $s=0$ of $\zeta(s,\mathbf{a}_{pq}) - \zeta(s,\mathbf{a}_{pq}^{\sigma})$ expressed explicitly as a square.

1. Sum over the multiplication-by-p relations starting at $i = 1,\ldots,\frac{p-1}{2}$, and multiplication-by-q relations starting at $j = 1,\ldots,\frac{q-1}{2}$.

2. Shift any terms that exceed $\frac{N-1}{2}$ back into the 1 to $\frac{N-1}{2}$ range and let A be the product of these sines.

3. Apply σ to the sum in step 1.

4. Shift this conjugated version of the sum back into the 1 to $\frac{N-1}{2}$ range and let B be the product of these sines.

Then the result is $\sin(\mathbf{a}_{pq})/\sin(\mathbf{a}_{pq}^{\sigma}) = B^{-2}A^2$. In terms of zeta functions this says that the coefficient of s near $s=0$ of $\zeta(s,\mathbf{a}_{pq}) - \zeta(s,\mathbf{a}_{pq}^{\sigma})$ is $2\log(A/B)$.

Chapter 3

The Imaginary Quadratic Case

3.1 Introduction

In this chapter we focus on both the imaginary quadratic analog of the results in the cyclotomic case and the subtle interplay between the two settings. We set our definitions to avoid ambiguity, review properties of the Siegel group, and give an explicit formula for a square root analogous to Anderson's square roots in the cyclotomic setting. Finally, we prove the theorem that if the square root of a modular unit has a level then that level is twice the level of the modular unit itself.

Let $\mathcal{H} = \{z \in \mathbb{C} | \operatorname{Im}\{z\} > 0\}$ denote the complex upper half plane; let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$ be the extended upper half plane and $\hat{\mathbb{C}}$ the compactified complex plane. Let Γ denote the (inhomogeneous) modular group or the group of all fractional linear transformations mapping $\mathcal{H}$ to itself. Γ is identified with the matrix group

$$\Gamma = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} |ad - bc = 1\}/\{\pm I\},$$

which is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

A map $f : \mathcal{H}^* \to \hat{\mathbb{C}}$ is called a *modular functions* (of level one) if

1. f is meromorphic in $\mathcal{H}$,
2. $f(A \circ z) = f(z)$ for all $A \in \Gamma$ and $z \in \mathcal{H}^*$,
3. there is an $a > 0$ so that for $\operatorname{Im}\{z\} > a$, $f(z)$ has an expansion in the local variable at $i\infty$, $q = e^{2\pi i z}$ of the form

$$f(z) = \sum_{n \geq n_0} a_n q^n, \ n \in \mathbb{Z}, \ a_{n_0} \neq 0.$$

so n_0 determines the behavior of f as $z \to \infty$. If $n_0 < 0$ then $f(i\infty) = \infty$; if $n_0 = 0$ then $f(i\infty = a_0$; and if $n_0 > 0$ then $f(i\infty) = 0$. In the last case, we call f a *cusp form.*

Fix a natural number $N > 2$. Let $\Gamma(N) \leq \Gamma$ be the (inhomogeneous) principal congruence subgroup modulo N, or the kernel of the reduction mod N map. In other words,

$$1 \longrightarrow \Gamma(N) \longrightarrow \Gamma \longrightarrow PSL_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1$$

is a short exact sequence.

By convention, we take $\Gamma(1) = \Gamma$. The upper half-plane modulo the action of $\Gamma(1)$ (written, by abuse of notation, $\mathcal{H}/\Gamma(1)$) is a singular surface with a one-point compactification, by the image of the point $i\infty$ under the stereographic projection, which makes it homeomorphic to the Riemann sphere. The completed non-singular curve is denoted $X(1)$. Similarly, $\mathcal{H}/\Gamma(N)$ can be compactified by adding finitely many points, the *cusps* of $\Gamma(N)$, or the translates of $i\infty$ under a full set of coset representatives for $PSL_2(\mathbb{Z}/N\mathbb{Z})$ in Γ. In this case the curve is denoted $X(N)$.

If H is a finite index subgroup in Γ the cusps, or translates of $i\infty$ under a full set of coset representative for H in Γ will hereafter be denoted $C(H)$.

3.2 Modular functions and the group of modular units

3.2.1 Modular functions

A *modular function* for a congruence subgroup $\Gamma(N)$ is a function, $f(z) : \mathcal{H}^* \to \hat{\mathbb{C}}$ such that

1. f is meromorphic in $\mathcal{H}$,

2. $f(A \circ z) = f(z)$ for all $A \in \Gamma(N)$ and $z \in \mathcal{H}^*$,

3. $f(z)$ has an expansion at each of the cusps in the local variable q of the form

$$f(z) = \sum_{n \geq n_0} a_n q^n, \; n \in \mathbb{Z}, a_{n_0} \neq 0.$$

If f is modular for $\Gamma(N)$, we say f has *level* N. A modular function of level N descends to a well-defined holomorphic function on $X(N)$. As before, if $n_0 > 0$ for all $\alpha \in C(\Gamma(N))$ then $f(z)$ is called a cusp form for $\Gamma(N)$.

An *almost modular function* of level N is one that is level N modulo constants. That is, for any $A \in \Gamma(N)$, we have $f(A \circ z) = c_A f(z)$ for some $c_A \in \mathbb{C}$; f is holomorphic at the cusps; but f may have Fourier coefficients that are not rational or even in $\mathbb{Q}(e^{\zeta N})$.

3.2.2 The full tower of modular functions $\mathcal{F}$

The set of modular functions invariant under the full modular group Γ is, in fact, a function field of genus one and is generated over $\mathbb{Q}$ by the classical j-function,

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3.$$

We write $\mathcal{F}_1 = \mathbb{Q}(j(z))$ and note that $\mathcal{F}_1$ is the full field of rational functions on $X(1)$ with rational Fourier coeffients. The j-function is normalized so that its q-expansion at $i\infty$ (which is the only cusp of $\mathcal{H}/\Gamma$) has integral coefficients. Thus, it is reasonable to define the ring of integers in this field to be $\mathbb{Z}[j]$.

Furthermore, the set of level N functions together with the N^{th} roots of unity generate a field extension of $\mathcal{F}_1$, denoted $\mathcal{F}_N$, which is a finite Galois extension of $\mathcal{F}_1$ with Galois group $PGL_2(\mathbb{Z}/N\mathbb{Z}) \cong \Gamma/\Gamma(N) \times (\mathbb{Z}/N\mathbb{Z})^\times$. The Galois action is given by thinking of $PGL_2(\mathbb{Z}/N\mathbb{Z}) \cong PSL_2(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})^\times$ and letting $PSL_2(\mathbb{Z}/N\mathbb{Z})$ act as usual as fractional linear transformations on z. Then the determinant of a matrix in $PGL_2(\mathbb{Z}/N\mathbb{Z})$ will be some $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and hence acts on the roots of unity of order dividing N by raising to powers via

$$d \mapsto \sigma_d : \zeta \mapsto \zeta^d.$$

In other words, if f has Fourier expansion

$$f(z) = \sum_{n \geq n_0} a_n q^n$$

then an elements of the form $A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in PGL_2(\mathbb{Z}/N\mathbb{Z})$ act as follows:

$$f(A \circ z) = \sum_{n \geq n_0} \sigma_d(a_n) q^n$$

and more general elements A of determinant d act as

$$f(A \circ z) = \sum_{n \geq n_0} \sigma_d(a_n)(A' \circ q)^n$$

where $A' = \frac{1}{d}A$.

Taking the integral closure of $\mathbb{Z}[j]$ in $\mathcal{F}_N$, we get a ring R_N, whose units, U_N, are the *modular units of level N*. It is not uncommon, however, to extend scalars to $\mathbb{C}$, that is, to study $U_N \otimes \mathbb{C} \subseteq R_N \otimes \mathbb{C}$. In this setting the set of functions with multiplicative inverses coincide precisely with the set of function whose divisor of zeros and poles is supported at the cusps of $X(N)$.

Finally, the compositum of the $\mathcal{F}_N$ over all N is called the *full tower of modular functions* $\mathcal{F}$ and we will refer to the units in the full tower of modular functions to mean the direct limit of the U_N) with respect to the natural inclusion maps and write

$$U = \lim_{\rightarrow} U_N.$$

3.3 The Siegel group

The fundamental functions we work with are the Siegel function,$\phi_\gamma(z)$. Among the myriad ways of defining the Siegel functions, we give here the Jacobi triple product expansion

$$\phi_\gamma(z) = -e^{\pi i \gamma}\theta_1(z)/\eta(z) = \zeta(1 - e^{2\pi i \gamma}) \prod_{m \geq 1} (1 - e^{2\pi i(mz+\gamma)})(1 - e^{2\pi i(mz-\gamma)})$$

where z is in the complex upper half-plane, $\zeta = -e^{\pi i(B_2(u)z + v(u-1))}$, $B_2(x) = x^2 - x + \frac{1}{6}$ and $\gamma = uz + v$ for $u, v \in (0, 1)$. From this expression for ϕ one can see that these functions generalize the square of Dedekind's eta function.

If u and v are in $\frac{1}{N}\mathbb{Z}$ then $\phi_\gamma(z)$ is level $12N^2$ and its $12N^{th}$ power is level N. In fact, the functions ϕ_γ^{12N} generate $\mathcal{F}_N$ over $\mathbb{Q}(\zeta_N)$.

3.3.1 The Kronecker limit formulae

In [Sie61], Siegel gives the transformation of the classical eta function by way of the first Kronecker limit formula.

$$\eta(z) = e^{2\pi iz/24}\prod_{m=1}^{\infty}(1-e^{2\pi imz}) = (2\pi)^{-12}\Delta(z)$$

(where $\Delta(z)$ is the classical discriminant function of the theory of elliptic functions). First we must define the real analytic Eisenstein series

$$E(z,s) = \sum_{m,n}\frac{y^s}{|mz+n|^{2s}}$$

where $z = x+iy$, m and n are not both zero, and $Re\{s\} > 1$. Siegel proves the transformation via Kronecker's first limit formula at $s = 1$

$$E(z,s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log(2) - \log(\sqrt{y}|\eta(z)|^2)) + O(s-1).$$

where γ is the Euler-Mascheroni constant.

Letting $G(s) = \pi^{-s}\Gamma(s)\sum_{m,n}|m\mu + n\nu + \alpha|^{-2s}$ for $\alpha = \nu\gamma = \nu(uz+v)$, Stark shows in [Sta80] that the second Kronecker limit formula gives an analogous result

$$G(0) = -2\log(|\phi_{(u,v)}(\mu/\nu)|)$$

where we see the function $\phi_{(u,v)}(z)$ cropping up in the analogous position as $\eta(z)^2$.

3.4 The Work of Daniel Kubert

In his papers [Kub79a] and [Kub79b] Kubert shows that the Siegel units generate the full group of units in R_N when $N = p^r$ for any prime p. He also shows that the Siegel units generate the whole modular unit group in $\mathcal{F}$ up to 2-cotorsion. If U is the group of units in $\mathcal{F}$ and S is the group generated by the Siegel units in $\mathcal{F}$ then U/S is exponent two. He goes on to prove that the second $\mathbb{Z}/2\mathbb{Z}$-cohomology of the universal ordinary distribution is non-trivial, and injects into the group of modular units modulo the Siegel group. Hence, U/S is non-trivial and there are modular units that are not Siegel units. Because U/S is two-torsion, in fact, each of these modular units squares to an element in the Siegel group. This begs the question of what the elements are that are in U but not S. We write down *explicit* elements in S that are squares in U, but whose square roots are not in S.

3.5 Towards an explicit formula for a square unit

In order to find a set of elements in S with explicit square roots in $U \setminus S$, we define the auxiliary functions h_γ for $\gamma = uz + v$.

$$h_\gamma(z) = \prod_{m \geq 0} (1 - e^{2\pi i(mz+\gamma)})$$

3.5.1 The odd relations

The h_γ are not modular; however, we are interested in combinations of them that are. For instance, we readily recognize that

$$\begin{aligned}\zeta(1 - e^{-2\pi i\gamma})^{-1} h_\gamma h_{-\gamma} &= \zeta(1 - e^{-2\pi i\gamma})^{-1} \prod_{m \geq 0} (1 - e^{2\pi i(mz+\gamma)})(1 - e^{2\pi i(mz-\gamma)}) \\ &= \zeta(1 - e^{2\pi i\gamma}) \prod_{m \geq 1} (1 - e^{2\pi i(mz+\gamma)})(1 - e^{2\pi i(mz-\gamma)}) = \phi_\gamma(z).\end{aligned}$$

This should be viewed as analogous to the functional equation for the classical Γ-function given by

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

The analog is clearer if one observes that the above identities (one for each γ), hereafter referred to as the odd relations for reasons to be explained later, can be reformulated in the following useful way:

$$\phi_\gamma(z) = \zeta(1-e^{-2\pi i\gamma})^{-1}\prod_{m\geq 0}(1-e^{2\pi i(mz+\gamma)})(1-e^{2\pi i(mz-\gamma)}) =$$
$$\zeta(1-e^{-2\pi i\gamma})^{-1}\prod_{m\geq 0}(1-e^{2\pi i(mz+\gamma)})(1-e^{2\pi i((m-u)z-v)}) =$$
$$\zeta(1-e^{-2\pi i\gamma})^{-1}\prod_{m\geq 0}(1-e^{2\pi i(mz+\gamma)})(1-e^{2\pi i((m-1+(1-u))z+(1-v))}) =$$
$$\zeta\prod_{m\geq 0}(1-e^{2\pi i(mz+\gamma)})(1-e^{2\pi i((m+(1-u))z+(1-v))}) =$$
$$\zeta h_\gamma h_{1-\gamma}$$

where by abuse of notation we define $1-\gamma$ to be $(1-u)z+(1-v)$.

3.5.2 The distribution relations

Das and Anderson use this Γ-relation together with the weight p and q distribution relations to produce new so-called algebraic Γ-monomials ($\mathbf{\Gamma}(\mathbf{a})$ for $\mathbf{a}\in U^-$, the universal odd distribution. Anderson then finds combinations of these monomials that have a square root, which allows him to give explicit generators for the maximal exponent two extension of $\mathbb{Q}^{ab}$ whose Galois group over $\mathbb{Q}^{ab}$ is a central extension of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$.

The analogy with the Siegel functions remains strong, but the formula for the weight p distribution relation is a bit more involved. Algebraicity (even over $\mathbb{Q}(j)$) is no longer the correct criterion, but rather modularity. As mentioned before h_γ itself is not modular; however, the above relation given by expressing ϕ_γ in terms of h_γ is an example of a modular relation.

We temporarily repress the notation that the product is over $m\geq 0$. Taking $\zeta_p = e^{\frac{2\pi i}{p}}$, a primitive p^{th} root of unity and $\gamma(r,s) = \gamma + \frac{r}{N}z + \frac{s}{N}$ for $N = pq$ we see that

$$\prod_{0\leq r,s\leq p-1} h_{\gamma(r,s)} = \prod_{0\leq r,s\leq p-1}(1-e^{2\pi i((m+\frac{r}{p})z+\gamma+\frac{s}{p})}) = \prod_{0\leq r,s\leq p-1}(1-\zeta_p^s e^{2\pi i((m+\frac{r}{p})z+\gamma)}).$$

Then, via the cyclotomic identity coming from the factorization of $1-x^n$, we observe that the effect of taking the product over s is to raise the exponential function to the p^{th} power, whence

$$\prod_{0\leq r,s\leq p-1}(1-\zeta_p^s e^{2\pi i((m+\frac{r}{p})z+\gamma)}) = \prod_{0\leq r\leq p-1}(1-e^{2\pi i((pm+r)z+p\gamma)})$$

Reindexing by $l = pm + r$, which again runs through all integers greater than or equal to 0 since r ranges from 0 to $p-1$, we arrive at

$$\prod_{l\geq 0}(1-e^{2\pi i(lz+p\gamma)}) = \prod_{0\leq r,s\leq p-1} h_{\gamma(r,s)} = h_{p\gamma}$$

In summary, the multiplication-by-p modularity relations say

$$\prod_{0\leq r,s\leq p-1} h_{\gamma(r,s)}/h_{p\gamma} = 1$$

is trivially modular for any u and v, not both integral.

3.5.3 Explicit squares

Note that a translation of u or v by an integer simply changes $\phi_\gamma(z)$ by an N^{th} root of unity, and since we are interested in generating $\mathcal{F}_N$, which contains the cyclotomic field $\mathbb{Q}(\zeta_N)$, we want to consider the group generated by the Siegel functions indexed by (u,v) mod $\mathbb{Z}^2$.

We define the new functions, recalling that $\langle x\rangle$ is the fractional part of x and redefining $\gamma = \langle u\rangle z + \langle v\rangle$ and $\zeta = -e^{\pi i(B_2(u)z+\langle v\rangle(\langle u\rangle-1))}$ for $B_2(x) = \langle x\rangle^2 - \langle x\rangle + \frac{1}{6}$ being the second Bernoulli polynomial.

1. $h_\gamma(z) = \prod_{m\geq 0}(1-e^{2\pi i(mz+\gamma)}) = \prod_{m\geq 0}(1-e^{2\pi i((m+\langle u\rangle)z+\langle v\rangle)})$
2. $h_{1-\gamma} = \prod_{m\geq 0}(1-e^{2\pi i(mz+1-\gamma)}) = \prod_{m\geq 0}(1-e^{2\pi i((m+1-\langle u\rangle)z+1-\langle v\rangle)})$
3. $\phi_\gamma(z) = \zeta h_\gamma h_{1-\gamma}$

The transformation properties of $\phi_\gamma(z)$ are now as follows:

1. $\phi_\gamma(z)$ depends only on the class of (u,v) in $(\mathbb{Q}/\mathbb{Z})^2$,

2. $\phi_{(u,v)}(z+1) = e^{\pi i/6}\phi_{(u+v,v)}(z)$ as before,

3. $\phi_\gamma(-1/z) = e^{-\pi i/2}\phi_{(v,-u)}(z)$ as well, and

4. $\phi_{-\gamma}(z) = -\phi_\gamma(z)$.

We still have

$$\prod_{0\le r,s\le p-1} h_{\gamma(r,s)} = h_{p\gamma}$$

from which is clearly follows that

$$\prod_{0\le r,s\le p-1} h_{1-\gamma(r,s)} = h_{p(1-\gamma)}$$

for any prime p. Thus, to write down the multiplication by p distribution-like relation for $\phi_\gamma(z)$ we need only calculate

$$\prod_{0\le r,s\le p-1} \zeta(r,s)$$

where $\zeta(r,s) = -e^{\pi i(B_2(u+\frac{r}{p})z+\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1))}$.

$$\prod_{0\le r,s\le p-1} \zeta(r,s) = \prod_{0\le r,s\le p-1} -e^{\pi i(B_2(u+\frac{r}{p})z+\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1))} =$$
$$(-1)^{pq} \prod_{0\le r,s\le p-1} e^{\pi i B_2(u+\frac{r}{p})z} \prod_{0\le r,s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1)}$$

But because B_2 is a distribution of weight p,

$$\prod_{0\le r,s\le p-1} e^{\pi i B_2(u+\frac{r}{p})z} = \prod_{0\le r\le p-1} (e^{\pi i B_2(u+\frac{r}{p})z})^p = e^{\pi i B_2(pu)pz}$$

And now we split

$$\prod_{0\le r,s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1)}$$

into

$$\prod_{0\le r,s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1/2)}$$

and

$$\prod_{0\le r,s\le p-1} e^{\frac{-\pi i}{2}\langle v+\frac{s}{p}\rangle}.$$

Because $B_1(x) =< x > -1/2$ is a distribution

$$\prod_{0\le r,s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1/2)} = \prod_{0\le s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u\rangle-1/2)} =$$
$$e^{\pi ip(\langle u\rangle-1/2)/2}\prod_{0\le s\le p-1} e^{\pi i(\langle v+\frac{s}{p}\rangle-1/2)(\langle u\rangle-1/2)} =$$
$$e^{\pi ip(\langle u\rangle-1/2)/2}e^{\pi i(\langle v\rangle-1/2)(\langle u\rangle-1/2)}.$$

Now returning to the last piece of the product, we compute

$$\prod_{0\le r,s\le p-1} e^{\frac{-\pi i}{2}\langle v+\frac{s}{p}\rangle} = \prod_{0\le r,s\le p-1} e^{\frac{-\pi i}{2}(\langle v+\frac{s}{p}\rangle-1/2)}e^{\frac{-\pi i}{4}} = e^{\frac{-\pi i}{2}(\langle v\rangle-1/2)}$$

so

$$\prod_{0\le r,s\le p-1} e^{\pi i\langle v+\frac{s}{p}\rangle(\langle u+\frac{r}{p}\rangle-1)} = e^{\pi ip(\langle u\rangle-1/2)/2}e^{\pi i(\langle v\rangle-1/2)(\langle u\rangle-1)}$$

So, in total, we have

$$\prod_{0\le r,s\le p-1} \zeta(r,s) = (-1)^{pq}e^{\pi iB_2(pu)pz}e^{\pi ip(\langle u\rangle-1/2)/2}e^{\pi i(\langle v\rangle-1/2)(\langle u\rangle-1)}$$

Recall that $\phi_\gamma(z) = \zeta h_\gamma h_{1-\gamma}$ so

$$\prod_{0\le r,s\le p-1} \phi_{\gamma(r,s)}(z) = \prod_{0\le r,s\le p-1} \zeta(r,s)h_{\gamma(r,s)}h_{1-\gamma(r,s)} = h_{p\gamma}h_{p(1-\gamma)}\prod_{0\le r,s\le p-1} \zeta(r,s)$$
$$= \phi_{p\gamma}(z)(-1)^{pq}e^{\pi iB_2(pu)pz}e^{\pi ip(\langle u\rangle-1/2)/2}e^{\pi i(\langle v\rangle-1/2)(\langle u\rangle-1)}.$$

Because the second part is too messy to compute with, we simply deal with the functions h_γ to define our square root. In the future when we wish to evaluate these functions to get extensions of imaginary quadratic fields, a more careful analysis of this lead term will be in order.

In analog with the cyclotomic setting where we saw $\frac{\sin(\mathbf{a}_{pq})}{\sin(\mathbf{a}_{pq})^{\sigma}}$ was a square for every $\sigma \in G^{ab}$, here we define

$$h_{\gamma_{pq}} = \frac{\prod_{\gamma \in A} h_\gamma}{\prod_{r,s=0}^{(q-1)/1} h_{\gamma(r,s)}} \cdot \frac{\prod_{r,s=0}^{(p-1)/2} h_{\gamma(r,s)}}{\prod_{\gamma \in B} h_\gamma}$$

where, recall, $\gamma = \langle u \rangle z + \langle v \rangle = \langle \frac{a}{N} \rangle z + \langle \frac{b}{N} \rangle$ and now $A = \{\gamma | 1 \leq a \leq (q-1)/2,\ 1 \leq b \leq (q-1)/2\}$ and similarly $B = \{\gamma | 1 \leq a \leq (p-1)/2,\ 1 \leq b \leq (p-1)/2\}$.

Then we define $H_{pq} = h_{\gamma_{pq}} h_{1-\gamma_{pq}}$ in analog with $\hat{\zeta}(s, \{x\}) = \zeta(s, \{x\}) + \zeta(s, \{-x\})$. For $\sigma \in PSL_2(\mathbb{Z}/N\mathbb{Z})$ then $\frac{H_{pq}}{H_{pq}^{\sigma}}$ is a square.

3.6 On the level of a square root

Theorem 3.6.1 (Beeson, 2009). *If $f(z) \in \mathcal{F}_N \setminus \mathcal{F}_N^2$ is a modular unit and $\sqrt{f(z)}$ has level M for some $M \in \mathbb{N}$ with $N|M$ then, in fact, $\sqrt{f(z)}$ has level $2N$.*

Proof. We assume $\sqrt{f(z)}$ is known to be invariant under the subgroup $\Gamma(M) \subseteq \Gamma(N)$. We will show that $\sqrt{f(z)}$ is invariant under $\Gamma(2N)$.

Let Γ_1 be the subgroup of $\Gamma(N)$ that fixes $\sqrt{f(z)}$. That is, for all $A \in \Gamma_1$ and $z \in \mathcal{H}^*$

$$\sqrt{f(A \circ z)} = \sqrt{f(z)}.$$

Because $\mathcal{F}_N(\sqrt{f(z)})$ is a quadratic extension of $\mathcal{F}_N$, the index $[\Gamma(N) : \Gamma_1] = 2$, and, thus, Γ_1 is a finite index subgroup of Γ as well. Furthermore, because $\sqrt{f(z)}$ is of level M, $\Gamma(M) \subseteq \Gamma_1$. So we have the linear ordering of fields below.

$$\begin{array}{c} \mathcal{F} \\ \Big| \}\Gamma(M) \\ \mathcal{F}_M \\ \Big| \\ \mathcal{F}^{\Gamma_1} \\ \Big| \, 2 \\ \mathcal{F}_N \\ \Big| \\ \mathbb{Q}(j(z)) \end{array}$$

Let $\mathcal{D}$ be a fundamental domain for Γ, that is, a simply connected subset of $\mathcal{H}^*$ such that $\mathcal{D}$ contains precisely one point from each Γ-orbit. If $\mathcal{D}_1$ is a fundamental domain for the subgroup of finite index $\Gamma_1 \subseteq \Gamma$ then it is made up of translates of $\mathcal{D}$ by a full set of coset representatives for Γ_1 inside Γ. Such a translate is called a *modular triangle*. Define the *fan width* n of a fundamental domain at the cusp α to be the order of the cyclic group that permutes the n Γ_1-inequivalent triangles meeting at α. Schoeneberg proves in [Sho74] that the conductor of a group Γ_1 is equal to the least common multiple of the fan widths at the rational cusps.

The index of $\Gamma(N)$ in Γ is the size of $SL_2(\mathbb{Z}/N\mathbb{Z})/\pm I$, which is

$$l = [\Gamma : \Gamma(N)] = \frac{1}{2}N^3 \prod_{p|N}(1 - \frac{1}{p^2}).$$

As observed above, $[\Gamma(N) : \Gamma_1] = 2$ so if we let σ be the automorphism of $\mathcal{F}^{\Gamma_1}$ fixing $\mathcal{F}_N$ such that

$$\sigma(\sqrt{f(z)}) = -\sqrt{f(z)}$$

then $\Gamma(N)$ decomposes as the disjoint union

$$\Gamma(N) = \Gamma_1 \cup \sigma\Gamma_1.$$

And if $\{A_1, \ldots, A_l\}$ is a complete set of representative for $\Gamma/\Gamma(N)$ then $\{A_1, \ldots, A_l, \sigma A_1, \ldots, \sigma A_l\}$ is a complete set of representatives for Γ/Γ_1. Recalling our notation $C(\Gamma(N))$ for the cusps of $\mathcal{H}/\Gamma(N)$, we see that

$$C(\Gamma(N)) \subset C(\Gamma_1)$$

and if α is a cusp of Γ_1 then either $\alpha \in C(\Gamma(N))$ or else $\sigma\alpha \in C(\Gamma(N))$.

Choose $\{A_1, \ldots, A_l\}$ to be a complete set of representatives for $\Gamma/\Gamma(N)$ such that

$$\mathcal{D}_N = \cup_i A_i(\mathcal{D})$$

is a fundamental domain for $\mathcal{H}/\Gamma(N)$. Let $\mathcal{D}_1$ be a fundamental domain for $\mathcal{H}/\Gamma_1$.

Schoeneberg's theorem implies that the least common multiple of the fan widths for $\mathcal{D}_N$ is N. We will use this to show that, for any cusp α of Γ_1, the fan width of $\mathcal{D}_1$ at α divides $2N$ so, by Schoeneberg's theorem, the conductor of Γ_1 divides $2N$. Because $\sqrt{f(z)} \notin \mathcal{F}_N$, $\sqrt{f(z)}$ must have level $2N$.

Let α be a cusp of Γ_1. As observed above, either α or $\sigma\alpha$ is a cusp of $\mathcal{D}_N$. (We can assume, without loss of generality that we have chosen an appropriate translate of $\mathcal{D}_N$ so that this is true). In the case that $\beta = \sigma\alpha$ is a cusp of $\mathcal{D}_N$, multiplication by σ is a homeomorphism between a neighborhood of α and a neighborhood of β so it suffices to prove the result for the cusps of $\mathcal{D}_1$ that are also cusps of $\mathcal{D}_N$.

Lemma 3.6.2 (Beeson, 2009)**.** *If α is a cusp of Γ_1 and of $\Gamma(N)$ that is of fan width n for $\mathcal{D}_N$ then its width for $\mathcal{D}_1$ is n or $2n$.*

Proof. Recall $\Gamma_1 \subseteq \Gamma(N)$ so $\Gamma \setminus \Gamma(N) \subseteq \Gamma \setminus \Gamma_1$ and that the fan width n of $\mathcal{D}_1$ at the cusp α is the order of the cyclic group that permutes the n Γ_1-inequivalent triangles meeting at α.

If two triangles are Γ_N-inequivalent then they are Γ_1-inequivalent so, assuming the width for $\mathcal{D}_N$ is n, the width for Γ_1 is at least n. Then since $[\Gamma(N) : \Gamma_1] = 2$, we see that the width of a triangle for Γ_1 is no more than $2n$. □

□

Chapter 4

Conclusion

4.1 Concluding remarks

We were inspired by Anderson's statement in [And02] that

> The relations standing between the Main Formula, the index formulas of Sinnott, Deligne reciprocity, the theory of Fröhlich, the theory of Das, the theory of the group cohomology of the universal ordinary distribution and Stark's conjecture and its variants deserve to be thoroughly investigated. We have only scratched the surface here. Stark's conjecture is relevant in view of the well known expansion
>
> $$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$
>
> of the Hurwitz zeta function at $s = 0$" [1].

In Chapter Two we gave the following results relating Anderson's construction to the Stark conjectures.

1. In Section 2.7.1 we proved the vanishing of a certain class of character sums of Hurwitz zeta functions.
2. In Section 2.7.2 we proved a family of associated trigonometric identities.
3. In Section 2.7.3 we give an algorithm for finding an explicit expression for the square root of certain units.

We were also inspired by Anderson's remark that

> Perhaps there is an analogue of the Main Formula over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing [And02].

In Chapter Three we treated the imaginary quadratic setting analog, where we have the following results.

1. In Section 3.6 we prove that if a modular unit has a level then that level is no more than twice the level of the original modular unit.
2. In Section 3.5.3 we gave an explicit expression for the square root of certain combination of Siegel units analogous to the cyclotomic case.

4.2 Future work and question

My research currently has three branches:

1. Further investigate of implications in the imaginary quadratic setting.

2. Find a multiplicative basis for the subgroup of cyclotomic units with index class number in the full unit group of $\mathbb{Q}(\zeta_m)^+$.

3. Let $k = \mathbb{Q}(\zeta_m)^+$ and $G = \mathrm{Gal}(k/\mathbb{Q})$. Let S be the set containing the infinite prime of $\mathbb{Q}$ and all the finite primes dividing m. Find a $\mathbb{Z}[G]$-submodule of U_S, the S-units in k, call it C, that contains the cyclotomic numbers and is of index class number in U_S. Show that $\widehat{H}^{q-2}(G, X) \cong \widehat{H}^q(G, C)$, where X is the $\mathbb{Z}[G]$-module of degree zero divisors of primes of k supported at the primes dividing m.

To accomplish (1), I will compute the transformation of my proposed square root using software I am developing. There are three possibilities for the outcome. First, the square root could fail to be in U. Second, the square root could transform under $\Gamma(N)$ for some N, but fail to have cyclotomic coefficients. Third, the square root could transform under $\Gamma(N)$ for some N and have cyclotomic coefficients. Let the imaginary quadratic base field be k. In the third case, special values of the function would generate an abelian extension K/k and its square root would generate a quadratic extension of K that is abelian over k. In this case, there would exist units that are squares. Together with (2), this could be used to show certain class numbers are even. In the second case, the square root would generate quadratic extensions of the maximal abelian extension of k in analogy with Anderson's construction. In this case, the next step would be to investigate to what extent these square roots generate the maximal almost abelian extension of k.

For (2), I will start with $m = pq$, for p and q distinct odd primes. Let $k = \mathbb{Q}(\zeta_{pq})^+$. Sinnott proves in [9] that the cyclotomic units have index in the full group of units equal to the class number of k. I have been working on finding a multiplicative basis for these units with an eye towards proving certain class number are even using results of Anderson. I have used Stark's conjecture in conjunction with explicit group-ring determinants and successfully accomplished this in examples. There are two avenues that I plan to explore more thoroughly: using Tate's representation-theoretic reformulation of Stark's conjectures [11] to break up the group determinant into character components; and using Sinnott's proof to break the construction of a basis into steps correlated to his intermediate index calculations.

(3) is a natural question that arises in the theory of Tate sequences and was pointed out to me by Popescu. In the case that m is a prime power, the cyclotomic units, call them C_S, are already index class number in U_S. In this setting, C_S is isomorphic to $\mathbb{Z}[G]\epsilon$, where ϵ is the Stark unit in k. Furthermore, the group of divisors on primes above p and infinity, which we shall call Y_S, is simply $Y_S \cong \mathbb{Z} \oplus \mathbb{Z}[G]$. Thus, the short exact sequence

$$0 \to X_S \to Y_S \stackrel{deg}{\to} \mathbb{Z} \to 0$$

splits and we see that X_S is isomorphic to $\mathbb{Z}[G]$. Hence, both C_S and X_S are cohomologically trivial. I am currently working on the next case, namely $m = pq$.

Bibliography

[And99] G. Anderson. A double complex for computing the sign-cohomology of the universal ordinary distribution. *Contemp. Math.*, 224:1–27, 1999.

[And02] G. Anderson. Kronecker-weber plus epsilon. *Duke Math. Journal*, 114(3):439–475, 2002.

[Bas66] Hyman Bass. Generators and relations for cyclotomic units. *Nagoya Math. J.*, 27(2):401–407, 1966.

[Das00] P. Das. Algebraic gamma monomials and double coverings of cyclotomic elds. *Trans. Amer. Math. Soc.*, 352:3557–3594, 2000.

[Gra77] G. Gras. Classes d'ideaux des corps abeliens et nombres de bernoulli generalises. *Ann. Inst. Fourier (Grenoble)*, 27(1, ix):1–66, 1977.

[Hil43] Obituary notice for Hilbert, 1943.

[Kub79a] D. Kubert. The $\mathbb{Z}/2\mathbb{Z}$-cohomology of the universal ordinary disribution. *Bulletin Soc. Math. France*, 107:203–224, 1979.

[Kub79b] D. Kubert. The universal ordinary disribution. *Bulletin Soc. Math. France*, 107:179–202, 1979.

[Kub81a] D. Kubert. The square root of the siegel group. *Proc. London Math. Soc.*, 43(2):193–226, 1981.

[Kub81b] S. Lang Kubert, D. *Modular Units*, volume 224 of *Die Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1981.

[Lan90] S. Lang. *Cyclotomic Fields I and II.* Springer-Verlag, 2 edition, 1990.

[Sho74] B. Shoeneberg. *Elliptic Modular Functions: an Introduction*, volume 203 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1974. Translated from the German by J. R. Smart and E. A. Schwandt.

[Sie61] I. C. L. Siegel. Advanced topics in analytic number theory: Lectures at the tata institute of fundamental research, bombay, india, 1961.

[Sin78] W. Sinnott. On the stickelberger ideal and the circular units of a cyclotomic field. *Ann. of Math.*, 108(1):107–134, 1978.

[Sta80] H. Stark. L-functions at s = 1. iv. first derivatives at s = 0. *Adv. in Math.*, 35(2):197–235, 1980.

[Tat84] J. Tate. *Les conjectures de Stark sur les fonctions L dArtin en s = 0, edited by Dominique Bernardi and Norbert Schappacher*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston Inc., 1984.