Symmetries of rational functions arising in Ecalle's study of multiple zeta values

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Abstract

In Ecalle's theory of multiple zeta values he makes frequent use of certain properties that express symmetries of rational functions in several variables. We focus on the properties of push-invariance, circ-neutrality, and alternality. Ecalle states and uses several implications about the relations between these symmetries. In this paper we investigate two of these implications and prove two results: first, that push-invariance and circ-neutrality imply the first alternality relation, but not the more general alternality relations, and second, that alternality does, indeed, imply circ-neutrality.

1 Introduction

The *multiple zeta values* are the numbers

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},$$

for $n_i \in \mathbb{N}$ with $n_r > 1$. Here, r is the *depth* and $n = n_1 + \cdots + n_r$ is the *weight* of $\zeta(n_1, \dots, n_r)$, where $1 \leq r < n$. When the depth $r = 1$ these values are known as *single zeta values* and are nothing other than the special values of the Riemann zeta function $\zeta(n) = \sum_{0<k} \frac{1}{k^n}$. Very little is known about the algebraic nature of the special values of the Riemann zeta function (let alone

multiple zeta values). Euler knew already in 1735 [Eul75] that

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}$$

and that, more generally,

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^n}{2(2n)!} \in \pi^{2n}\mathbb{Q}.$$

A widely believed folklore conjecture, for example, states that the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(2n+1)$ are algebraically independent over $\mathbb{Q}$ for any integer $n \geq 1$.

In studying algebraic independence of zeta values one is led naturally to the study of similar questions for multiple zeta values. In this context it turns out to be useful to define what is called the $\mathbb{Q}$-algebra of formal multiple zeta values, which is generated by symbols of the form $Z(k_1, \ldots, k_r)$ modulo the standard regularized shuffle and stuffle algebraic relations. The structure of this algebra is a tantalizing and much-studied question. Several authors, particularly M. Hoffman [Hof97] and D. Zagier [Zag93] have made seminal contributions to its study. J. Ecalle has designed and implemented a vast program to study it, using his own personal language and theory, which yields beautiful and natural generalizations, restatements and proofs of some of the important facts and conjectures concerning multiple zeta values.

The key to his theory is to place the whole situation within a bigger universe, known as the theory of *moulds*; in this paper we consider only "moulds" which are in fact rational functions of several variables and which are the most relevant moulds in the study of multiple zeta values. An essential feature of Ecalle's theory is the study of symmetry properties of moulds, such as the properties of circ-neutrality[1], push-invariance and alternality (defined in Section 2) that we concentrate on in this paper. Ecalle's seminal article [Eca11] contains a grand survey of some of his main ideas. However, because of the length and depth of the theory, there are few proofs. The following assertion (for general moulds, but we restrict it to rational functions here) appears in section 11.9 of [Eca11].

Assertion 1.1. *If A is a rational function that is* push-*invariant and* circ-*neutral, then A is alternal.*

[1] Ecalle uses the terminology pus-neutrality for this property.

This statement turns out not to be true in full generality[2], but we prove that push-invariance and circ-neutrality do imply the first alternality relations in Section 4. Moreover, we found via Maple calculations that there is no counterexample to the more general case in two variables and low degree, but we found one in degree three with five variables, which we give in Section 4.

In section 2.4 of [Eca11] we find the following assertion (again for general moulds):

Assertion 1.2. *Every alternal rational function is* circ*-neutral.*

The main goal of Section 5 is to detail the proof of this assertion. Our strategy to prove Assertion 1.2 is to first reduce it to the polynomial case, which is done in Section 3. We can then treat the polynomial case using properties of certain Lie algebras and Ecalle's multiplication operator mu, whose definition we introduce in Section 5.

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2 Definitions

There are three properties of rational functions that are of interest: push-invariance, circ-neutrality, and alternality.

If $A(u_1, \ldots, u_r)$ is a rational function in r variables we define

$$\operatorname{circ} A(u_1, \ldots, u_r) = A(u_r, u_1, \ldots, u_{r-1})$$

and, for $u_0 = -u_1 - u_2 - \cdots - u_r$,

$$\operatorname{push} A(u_1, \ldots, u_r) = A(u_0, u_1, \ldots, u_{r-1}).$$

[2]However, the statement forms part of the proof of a result for which Ecalle gave a very different but complete proof in a subsequent paper.

Definition 2.1. *A is* push*-invariant if*

$$\operatorname{push} A = A.$$

Definition 2.2. *A is* circ*-neutral if*

$$A + \operatorname{circ} A + \operatorname{circ}^2 A + \cdots + \operatorname{circ}^{r-1} A = 0.$$

Definition 2.3. *A is alternal if*

$$\sum_{w \in sh(u_1 u_2 \ldots u_k)(u_{k+1} \ldots u_r)} A(w) = 0$$

for all $1 \leq k \leq \lceil \frac{r}{2} \rceil$, where $sh(w, w')$ is the set of all possible shuffles of the words w and w', that is, permutations of the letters of w and w' that preserve the ordering in w and the ordering in w'.

A more technical definition of the shuffle of two words is the following. Let $\mathcal{A}$ be the set of all words in the alphabet $\{a_1, \ldots, a_{r+s}\}$ and suppose $w = a_1 a_2 \cdots a_r$ and $w' = a_{r+1} a_{r+2} \cdots a_{r+s}$. Let the symmetric group on $r+s$ letters S_{r+s} act on $\mathcal{A}$ by permuting the indices. Let

$$T = \{\sigma \in S_{r+s} \,|\, \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r) \text{ and}$$

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s)\ \}$$

and note that if $x \leq r$ and $y > r$ there is no relation whatsoever imposed on $\sigma^{-1}(x)$ and $\sigma^{-1}(y)$. Then

$$sh(w, w') = \{w'' \in \mathcal{A} \,|\, w'' = \sigma(ww') \text{ for some } \sigma \in T\}.$$

Example 2.4. *An example of a polynomial that is* push*-invariant,* circ*-neutral, and alternal is $A(u_1, u_2) = -2u_1^3 - 3u_1^2 u_2 + 3u_1 u_2^2 + 2u_2^3$.*

Remark 2.5. *We note here that* circ*-neutrality and alternality are additive properties that respect the multi-degree of a polynomial. Hence, once we have reduced to the polynomial case in Section 3 we can further reduce to the case of monomials of fixed multi-degree.*

3 Reduction to the polynomial case

In this section, we reduce the study of Assertion 1.2 and similar questions to the case of polynomials. It turns out that to show that a family of rational functions satisfy circ-neutrality, push-invariance, or alternality, it suffices to show that a corresponding family of polynomials satisfies that property.

Lemma 3.1. *Every rational function $A(u_1, \ldots, u_r)$ can be written in the form $A = P/Q$ where P and Q are polynomials in $u_1, \ldots, u_r$ such that $Q(u_1, \ldots, u_r)$ is invariant under* push *and any permutation of $u_1, \ldots, u_r$. If $A = P/Q$ is such an expression for A then A is alternal (resp.* push*-invariant, resp.* circ*-neutral) if and only if P is alternal (resp.* push*-invariant, resp.* circ*-neutral).*

Proof. Consider a rational function $A \in \mathbb{Q}(u_1, \ldots, u_r)$. We note that

$$\mathrm{circ}^r = \mathrm{id} \;\text{ and }\; \mathrm{push}^{r+1} = \mathrm{id}\,.$$

Write $A = p/q$ with $p, q \in \mathbb{Q}[u_1, \ldots, u_r]$. Set $u_0 = -u_1 - \ldots - u_r$, and let the symmtric group S_{r+1} act on the $r+1$ indices $0, 1, \ldots, r$ by permutation. We define

$$Q(u_1, \ldots, u_r) = \prod_{\sigma \in S_{r+1}} q(u_{\sigma(1)}, \ldots, u_{\sigma(r)}).$$

Then the polynomial Q in r variables is push-invariant and invariant under any permutation of the r indices $1, \ldots, r$. Then we simply set

$$P(u_1, \ldots, u_r) = p(u_1, \ldots, u_r) \prod_{\sigma \in S_{r+1},\, \sigma \neq \mathrm{id}} q(u_{\sigma(1)}, \ldots, u_{\sigma(r)}),$$

so that $f = P/Q$, which is of the desired form.

Finally, because Q is invariant under push and any permutation of $u_1, \ldots, u_r$, we have that A is circ-neutral (resp. push-invariant, resp. alternal) if and only if P is circ-neutral (resp. push-invariant, resp. alternal), which proves the lemma. □

4 push-invariance and a counterexample to 1.1

Checking Assertion 1.1 in Maple [map13], one finds no counterexample for polynomials in fewer than three variables in degree five (or seven variables in degree four), but in 3 variables and degree 5, a nice counterexample appears; we give it at the end of this section. First, let us show that, even though push-invariance and circ-neutrality do not imply alternality in general, they do imply the first alternality relation (shuffling one variable into the rest).

Theorem 4.1. *If A is* circ-*neutral and* push-*invariant then the first alternal relation holds. That is,* $\displaystyle\sum_{w\in sh(u_1)(u_2,\ldots,u_r)} A(w) = 0.$

Proof. The circ-neutral relation guarantees that

$$A + \mathrm{circ}\, A + \mathrm{circ}^2 A + \cdots + \mathrm{circ}^{r-1} A = 0.$$

That is,

$$A(u_1, \ldots, u_r) + A(u_r, u_1, \ldots, u_{r-1}) + \cdots + A(u_2, u_3, \ldots u_r, u_1) = 0.$$

Now, assuming A is push-invariant, we have that A under any change of variables is push-invariant. It follows that $\mathrm{circ}^k A$ is push-invariant for all k. Thus, we can push each term an appropriate number of times, preserving equality, to get

$$\mathrm{push}\, A + \mathrm{push}^{r-1} \mathrm{circ}\, A + \mathrm{push}^{r-2} \mathrm{circ}^2 A + \cdots + \mathrm{push}^2 \mathrm{circ}^{r-1} A = 0.$$

So we see that

$$A(u_0, u_1, \ldots, u_{r-1}) + A(u_1, u_0, \ldots, u_{r-1}) + A(u_1, u_2, u_0, \ldots, u_{r-1}) + \cdots +$$
$$A(u_1, u_2, \ldots, u_0, u_{r-1}) + A(u_1, \ldots, u_{r-1}, u_0) = 0,$$

which is precisely the first alternal relation, written in terms of the variables $u_0, u_1, \ldots, u_{r-2}, u_{r-1}$. Just as push-invariance implies push-invariance under any change of variable, so does knowing the first alternal relations with one set of variables imply the first alternal relations after any change of variables. This concludes the proof. □

Next we observe that Assertion 1.1 is trivially true for the case of linear forms in any given number of variables. Indeed, in the following lemma we show that there are no nonzero linear push-invariant forms at all.

Lemma 4.2. *There are no non-trivial* push-*invariant linear forms.*

Proof. Assume that the linear form $A(u_1, \ldots, u_r) = a_1 u_1 + \ldots + a_r u_r$ is push-invariant. Then the equation $A = \operatorname{push} A$ implies that

$$a_1 u_1 + \ldots + a_r u_r = a_1(-u_1 - \ldots - u_r) + a_2 u_1 + \ldots + a_r u_{r-1}.$$

We compare coefficents on both sides and obtain the system of linear equations

$$\begin{aligned} a_1 &= -a_1 + a_2 \\ a_2 &= -a_1 + a_3 \\ &\vdots \\ a_{r-1} &= -a_1 + a_r \\ a_r &= -a_1. \end{aligned}$$

If $a_r = 0$ then all the other a_i have to be zero. If a_r is nonzero then we may assume after normalization that $a_r = 1$. The last equation then gives that $a_1 = -1$; the first equation gives $a_2 = -2$; the second $a_3 = -3$, until we obtain from the penultimate equation that $a_r = -r$, which is a contradiction to $a_r = 1$. □

Remark 4.3. *Determining the dimension of the subspace of* push-*invariant polynomials is non-trivial. We include here Maple calculations of the dimension of the space of* push-*invariant,* circ-*neutral polynomials for small values of r (number of variables) and n (degree) [map13]:*

$r \vert^{n}$	1	2	3	4	5	6	7	8	9
2	0	0	1	0	1	1	1	1	2
3	0	1	0	2	1	3	2	5	3
4	0	0	1	1	3	3	6	7	11
5	0	1	2	3	5	11	14	24	34
6	0	0	1	3	8	14	28	–	–
7	0	1	2	8	13	31	55	–	–
8	0	0	3	5	19	43	–	–	–
9	0	1	2	11	29	–	–	–	–

Entering the rows individually into the On-line Encyclopedia of Integer Sequences (OEIS), we note that the dimensions for $r = 2$ appear to correspond to the dimensions $[(2n+2)/4] - [(2n+4)/6]$ of cusp forms of weight $2n+6$ on $\mathrm{SL}_2(\mathbb{Z})$. Apart from this the significance of these dimensions is not clear.

Despite push-invariance being a very strong property, it turns out, as observed at the beginning of this section, that push-invariance and circ-neutrality together are not enough to ensure that a given rational function is alternal in higher degree. The following polynomial P is the smallest counterexample that we found to Assertion 1.1. The Maple code used can be found in Appendix A.

$$\begin{aligned}P = &- u_1^2u_3 + u_1^2u_4 + 2u_1u_2u_4 - 2u_1u_2u_5 - u_1u_3^2 - 2u_1u_3u_4 + u_1u_4^2 + \\ &2u_1u_4u_5 + u_2^2u_3 - u_2^2u_5 + u_2u_3^2 + 2u_2u_3u_5 - 2u_2u_4u_5 - u_2u_5^2 - u_3^2u_4 + \\ &u_3^2u_5 - u_3u_4^2 + u_3u_5^2\end{aligned}$$

5 circ-neutrality and a proof of Assertion 1.2

The goal of this section is to prove that any alternal rational function is, in fact, circ-neutral.

Lemma 5.1. *To show any alternal rational function is* circ*-neutral, it suffices to show that any alternal polynomial is* circ*-neutral.*

Proof. Let $A(u_1, \ldots, u_r)$ be an alternal rational function. Then by Section 3, we can write $A = P/Q$ with P alternal and Q invariant under any permutation of the variables $u_1, \ldots, u_r$. Thus, to show A is circ-neutral, it suffices to show that P is. □

The remainder of this section is devoted to proving that if a polynomial P is alternal then it must also be circ-neutral (see Theorem 5.5).

Our approach is to first prove a proposition that gives a useful characterization of circ-neutral polynomials. We then show that an alternal polynomial must fit this characterization. In order to prove this proposition, we first introduce Ecalle's multiplication operator mu on two polynomials.

Definition 5.2. *If A and B are polynomials in r_A (resp. r_B) variables then $mu(A, B)$ is a polynomial in $r = r_A + r_B$ variables defined as*

$$\mathrm{mu}(A, B) = A(u_1, \ldots, u_{r_A})B(u_{r_A+1}, \ldots, u_{r_A+r_B}).$$

Furthermore, we set $[[A, B]] = \mathrm{mu}(A, B) - \mathrm{mu}(B, A)$.

Note that $\text{mu}(B, A) = B(u_1, \ldots, u_{r_B})A(u_{r_B+1}, \ldots, u_{r_A+r_B})$ is also a polynomial in r variables.

Proposition 5.3. *Let A and B be monomials of degree d_A (resp. d_B) in r_A (resp. r_B) variables. Then $M := [[A, B]]$ is* circ*-neutral.*

Proof. M is a homogeneous polynomial of degree $d = d_A + d_B$ in $r = r_A + r_B$ variables given by

$$M = A(u_1, \ldots, u_{r_A})B(u_{r_A+1}, \ldots, u_{r_A+r_B}) - B(u_1, \ldots, u_{r_B})A(u_{r_B+1}, \ldots, u_{r_A+r_B}).$$

Writing $A(u_1, \ldots, u_{r_A}) = u_1^{a_1} \cdots u_{r_A}^{a_{r_A}}$ and $B(u_1, \ldots, u_{r_B}) = u_1^{b_1} \cdots u_{r_B}^{b_{r_B}}$, we have

$$M(u_1, \ldots, u_r) = u_1^{a_1} \cdots u_{r_A}^{a_{r_A}} u_{r_A+1}^{b_1} \cdots u_r^{b_{r_B}} - u_1^{b_1} \cdots u_{r_B}^{b_{r_B}} u_{r_B+1}^{a_1} \cdots u_r^{a_{r_A}}.$$

If we now consider the sum

$$M + \operatorname{circ} M + \operatorname{circ}^2 M + \cdots + \operatorname{circ}^{r-1} M = \sum_{i=1}^{r} M(u_i, \ldots, u_r, u_1, \ldots, u_{i-1}),$$

we see that the positive monomial from the i-th term cancels with negative monomial in the $i + r_A$-th term (with indices taken modulo r in the set $\{1, \ldots, r\}$). Thus, in fact,

$$\sum_{i=1}^{r} M(u_i, \ldots, u_r, u_{r+1}, \ldots, u_{i-1}) = 0,$$

so M is indeed circ-neutral.

□

Remark 5.4. *It follows directly from Proposition 5.3 and additivity of* circ*-neutrality that if A and B are any polynomials, not necessarily monomials, then the polynomial $[[A, B]]$ is* circ*-neutral.*

Note that both alternality and circ-neutrality are empty conditions for a polynomial in one variable, so we adopt the convention of saying alternality implies circ-neutrality in this case. If A is a polynomial in $r > 1$ variables, because alternality and circ-neutrality both respect degree, we can assume that A is homogenous of multi-degree d. Additivity and Proposition 5.3 together imply that, in order to prove Assertion 1.2, it suffices to show that any alternal A is a linear combination of terms of the form $[[B, D]]$; this is the method we use to prove the following theorem.

Theorem 5.5. *If A is a homogeneous alternal polynomial of degree d in $r > 1$ variables then A is* circ*-neutral.*

Proof. We capitalize on properties of a certain Lie algebra to show that if A is alternal then it is of the desired form $[[B, D]]$.

Let $\mathbb{Q}[u_1, \ldots, u_r]$ be the ring of polynomials in r commuting variables and let $R = \mathbb{Q}\langle x, y\rangle$ be the ring of polynomials in two non-commuting variables. Let R_r denote the $\mathbb{Q}$ vector subspace of R spanned by monomials containing exactly r y's. Note that R_0 is spanned by the monomials that are powers in x, including $x^0 = 1$. As a $\mathbb{Q}$ vector space, R is the direct sum of the R_r's for all r.

For $r \geq 1$, define the map of $\mathbb{Q}$ vector spaces

$$\phi_r : \mathbb{Q}[u_1, \ldots, u_r] \longrightarrow R_r \tag{1}$$

by extending linearly from the map $u_1^{a_1} \cdots u_r^{a_r} \mapsto C_{a_1+1}C_{a_2+1} \ldots C_{a_{r-1}+1}C_{a_r+1}$, where

$$C_i = \mathrm{ad}(x)^{i-1}y = [x, \cdots x, [x, [x, y]]] \cdots],$$

and $[x, y] = xy - yx$ is the standard Lie bracket. So, for example, we have

$$\begin{aligned}
\phi_r(1) &= \phi_r(u_1^0 \cdots u_r^0) = C_1^r = y^r,\\
\phi_r(u_1) &= \phi_r(u_1^1 u_2^0 \cdots u_r^0) = C_2 C_1^{r-1} = [x, y]y^{r-1} = (xy - yx)y^{r-1},\\
\phi_r(u_1^2) &= \phi_r(u_1^2 u_2^0 \cdots u_r^0) = C_3 C_1^{r-1} = [x, [x, y]]y^{r-1},\\
\phi_r(u_2) &= \phi_r(u_1^0 u_2^1 u_3^0 \cdots u_r^0) = C_1 C_2 C_1^{r-1} = y[x, y]y^{r-2},\\
\phi_r(7u_1^2 - 5u_2) &= \phi_r(7u_1^2 u_2^0 \cdots u_r^0 - 5u_1^0 u_2^1 u_3^0 \cdots u_r^0) = 7C_3 C_1^{r-1} - 5C_1 C_2 C_1^{r-2}, \text{ and}\\
\phi_r(u_2^2 u_r) &= \phi_r(u_1 u_2^2 u_r) = C_1 C_3 C_1^{r-3} C_2.
\end{aligned}$$

A key observation is that if B and D are two polynomials in s and t variables, respectively, then we have $\phi_{s+t}(\mathrm{mu}(B, D)) = \phi_s(B)\phi_t(D)$, so that

$$\begin{aligned}
\phi_{s+t}(\mathrm{mu}(B, D) - \mathrm{mu}(D, B)) &= \phi_s(B)\phi_t(D) - \phi_t(D)\phi_s(B)\\
&= [\phi_s(B), \phi_t(D)],
\end{aligned} \tag{2}$$

where mu is the operator from Definition 5.2 above.

Let $\mathrm{Lie}[x, y]$ be the free Lie algebra generated by x and y under the Lie bracket, $[f, g] = fg - gf$, and let $\mathrm{Lie}_r[x, y]$ be the $\mathbb{Q}$ vector subspace of $\mathrm{Lie}[x, y]$ spanned by brackets of exactly r y's with any number of x's, i.e., the space of Lie polynomials with homogeneous degree r in y.

Lemma 5.6. *If A is an alternal polynomial in r variables then $\phi_r(A)$ lies in* $\mathrm{Lie}_r[x,y]$.

Proof. We note that if A is an alternal polynomial in the variables $u_1,\ldots,u_r$, then the polynomial $\phi_r(A)$ considered as a polynomial in the variables C_i satisfies the shuffle relations. By Theorem 1.4 (Section 1.5) in [Rau93] any polynomial that satisfies the shuffle relations in the variables C_i is a Lie polynomial in the variables C_i. Now we recall that C_i is defined as

$$C_i = \mathrm{ad}(x)^{i-1}(y),$$

and, hence, is a Lie polynomial in the variables x and y. Thus, any Lie polynomial in the variables C_i is also a Lie polynomial in the variables x and y.

□

Let $\mathcal{A}l_r$ denote the subspace of alternal polynomials in $\mathbb{Q}[u_1,\ldots,u_r]$.

Lemma 5.7. *The map $\phi_r : \mathcal{A}l_r \longrightarrow \mathrm{Lie}_r[x,y]$ is a bijection for every $r \geq 1$.*

Proof. The main point is that the C_i are algebraically independent. To see this we first observe that by Lazard elimination we have a direct sum $\mathrm{Lie}[x,y] = \mathrm{Lie}[x] \oplus \mathrm{Lie}[C_1, C_2, \ldots]$ (see Proposition 10 a) in [Bou]). Part b) of the same proposition proves that if $C_1', C_2', ...$ are non-commutative indeterminates, then the map induced by $C_i' \mapsto C_i$ is an isomorphism from the free Lie algebra $\mathrm{Lie}[C_1', C_2', ...]$ to $\mathrm{Lie}[C_1, C_2, ...]$. Hence the latter is a free Lie algebra. Finally, Theorem 1 (b) in §3, no. 1, of [Bou] shows that the universal enveloping algebra of $\mathrm{Lie}[C_1', C_2', ...]$ is just the free non-commutative polynomial algebra $\mathbb{Q}\langle C_1', C_2', ...\rangle$, and the previous isomorphism extends uniquely to an isomorphism of universal enveloping algebras. This shows that the universal enveloping algebra of $\mathrm{Lie}[C_1, C_2, ...]$, namely the polynomial algebra $\mathbb{Q}\langle C_1, C_2, ...\rangle$ is also free on these variables, showing that they are indeed algebraically independent. Thus, any element of $\mathrm{Lie}[C_1, C_2, \ldots]$ has a unique expression as a polynomial in the C_i, and thus a unique preimage under the map (1). So the map ϕ_r is both injective and surjective, which concludes the proof. □

Remark 5.8. *The above lemma shows that the map $\psi_r : \mathrm{Lie}_r[x,y] \longrightarrow \mathcal{A}l_r$, which is the restriction to the Lie algebra of the map on $\mathbb{Q}\langle C_1, C_2, \ldots\rangle$ defined by $\psi_r(C_{a_1}...C_{a_r}) = u_1^{a_1-1}...u_r^{a_r-1}$, is an explicit inverse to ϕ_r.*

Lemma 5.9. *Let $r > 1$. Then any element of* $\mathrm{Lie}_r[x, y]$ *can be written as a sum* $\sum_i a_i[f_i, g_i]$ *with* $f_i, g_i \in \mathrm{Lie}[C_1, C_2, \ldots]$ *for all* i.

Proof. Note that the assertion of the Lemma is equivalent to saying that one can decompose any element of $\mathrm{Lie}_r[x, y]$ into a sum of brackets in which none of the f_i, g_i is equal to x.

We observe that every element in $\mathrm{Lie}_r[x, y]$ for $r > 1$ can be written as a linear combination of Lie brackets of r y's and any number of x's. By additivity it is hence enough to prove the lemma for a single Lie bracket of r y's and s x's. For this we note that any Lie bracket is of the form $[f, g]$ with $f, g \in \mathrm{Lie}[x, y]$. If both f and g are themselves Lie brackets, or if f or g is equal to y, then we are already in the desired form. Hence it remains to consider the case where f or g is equal to x. Without loss of generality we may assume that $f = x$. Thus we have reduced the proof to showing that a Lie bracket of the form $[x, g]$ can be rewritten in the form $\sum_i a_i[f_i, g_i]$ with none of the f_i, g_i equal to x.

We prove this claim by induction on the degree of the bracket, which equals $r + s$ in the notation above. Recall that we have assumed $r > 1$ and hence $r + s \geq 4$, since we are considering brackets of the form $[x, g]$ where g is a Lie bracket containing at least 2 y's, so of degree at least 3. The base case is thus the example $g = [y, [x, y]]$.

To write this as a linear combination $\sum_i a_i[f_i, g_i]$ with none of the f_i, g_i equal to x, we use the Jacobi relation

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0,$$

with $a = x$, $b = y$ and $c = [x, y]$. This yields

$$[x, [y, [x, y]] + [[x, y], [x, y]] + [y, [[x, y], x]] = 0.$$

The middle term is zero, so we can rewrite $[x, g] = [x, [y, [x, y]] = -[y, [[x, y], x]]$, which is of the desired form. This completes the base case of the induction.

Next we assume that all Lie brackets up to total degree $r + s - 1$ in x and y can be written in the form stated in the lemma. Consider a Lie bracket of the form $[x, g]$ in degree $r + s$. Then g is a Lie bracket of degree $r + s - 1$, and hence by our induction hypothesis we can write $g = \sum_i a_i[f_i, g_i]$ with none of the f_i, g_i equal to x. Now we use again the Jacobi relation to rewrite the problematic Lie bracket as

$$[x, g] = \sum_i a_i[x, [f_i, g_i]] = -\sum_i a_i[g_i, [x, f_i]] - \sum_i a_i[f_i, [g_i, x]].$$

Since none of the f_i and the g_i are equal to x, this completes the proof of the lemma. □

We can now complete the proof of Theorem 5.5.

Let A be an alternal polynomial in $r > 1$ variables. Then $\phi_r(A)$ is in $\mathrm{Lie}_r[x, y]$ by Lemma 5.6. Thus, by Lemma 5.9, we can write

$$\phi_r(A) = \sum_i a_i[f_i, g_i]$$

where none of the f_i or g_i is equal to x. For each i in the sum, we thus have $f_i \in \mathrm{Lie}_{s_i}[x, y]$ for some $s_i \geq 1$, and $g_i \in \mathrm{Lie}_{t_i}[x, y]$ for some $t_i \geq 1$. Now, by Lemma 5.7, ϕ_{s_i} and ϕ_{t_i} are surjective, so there exist polynomials B_i and D_i satisfying $\phi_{s_i}(B_i) = f_i$ and $\phi_{t_i}(D_i) = g_i$.

Summing this up and using (2) and the linearity of ϕ_r, we have

$$\begin{aligned}
\phi_r(A) &= \sum_i a_i[f_i, g_i] \\
&= \sum_i a_i[\phi_{s_i}(B_i), \phi_{t_i}(D_i)] \\
&= \sum_i a_i \phi_{s_i+t_i}(\mathrm{mu}(B_i, D_i) - \mathrm{mu}(D_i, B_i)) \\
&= \phi_r\left(\sum_i a_i[[B_i, D_i]]\right).
\end{aligned}$$

Finally, by the injectivity of ϕ_r proved in Lemma 5.7, we have that

$$A = \sum_i a_i[[B_i, D_i]].$$

By Proposition 5.3 and additivity this is enough to guarantee that A is circ-neutral, which concludes the proof of Theorem 5.5.

□

A Appendix A

This is the Maple code for creating a generic push-invariant, circ-neutral polynomial of degree n in r variables and checking whether it fails on the second alternality relation. One inputs the degree n of the polynomial and the number r of variables at the beginning of the program as can be seen below. We used this code to obtain the counterexample to Assertion 1.1 in section 4.

```
#
#(1) Create arbitrary polynomial in r variables of degree n
#(2) Solve the systems for push-invariant and circ-neutral
#(3) Check whether polynomial satisfies second alternality
# relation
n:=3:
r:=5:
#procedure to compute push(P)
pushP:=proc(P,r)
  local Q:
  Q:=expand(subs({u[1]=-add(u[k],k=1..r),seq(u[k]=u[k-1],
  k=2..r)},P)):
  return Q:
end proc:
#procedure to compute circ(P)
circP:=proc(P,r)
  local Q:
  Q:=expand(subs({u[1]=u[r],seq(u[k]=u[k-1],k=2..r)},P)):
  return Q:
end proc:
U:=[seq(u[i],i=1..r+1)]:
with(combinat):
#Creation of generic polynomial of deg n in r vars with
#indeterminate coefficients
C1:=partition(n+r):
C2:=[]:
for i from 1 to nops(C1) do
  if(nops(C1[i])=r) then C2:=[op(C2),op(permute(C1[i]))] fi:
od:
C3:=[]:
for i from 1 to nops(C2) do
  C3:=[op(C3),[seq(C2[i][j]-1,j=1..r)]]
od:
```

```
P:=0:
for i from 1 to nops(C3) do
  P:=P+a[i]*product(u[j]^C3[i][j],j=1..r):
od:
#That's it, P is the generic polynomial
#Now make linear system for the circ-neutrality relation
#P+circ(P)+...+circ^(r-1)(P)=0
PP[0]:=P:
for i from 1 to r-1 do PP[i]:=circP(PP[i-1],r): od:
Q:=add(PP[i],i=0..r-1):
COEFFScircneut:={coeffs(Q,U)}:
#Now make linear system for push-invariance
Q:=expand(P-pushP(P,r)):
COEFFSpushinv:={coeffs(Q,U)}:
Sols:=solve(COEFFScircneut union COEFFSpushinv):
P:=expand(subs(Sols,P)):
print("generic push-invariant, circ-neutral polynomial of
degree",n,"in", r,"variables"):
print(P):
#Test whether it satisfies the second alternality relation
#corresponding to sh((1,2),(3,..,r))
C4:=choose(r,2):
#build shuffle permutations
for i from 1 to binomial(r,2) do
    shu[i]:=[seq(m,k=1..r)]:
    shu[i][C4[i][1]]:=1:
    shu[i][C4[i][2]]:=2:
    cc:=3:
    for k from 1 to r do
      if(shu[i][k]=m) then shu[i][k]:=cc: cc:=cc+1: fi:
    od:
od:
#build shuffle relation
Q:=0:
for i from 1 to binomial(r,2) do
    Q:=Q + subs({seq(u[k]=u[shu[i][k]],k=1..r)},P)
od:
Q:=expand(Q):
```

```
if(Q=0) then print("satisfies second shuffle relation") fi:
if(Q<>0) then print("fails second shuffle relation") fi:
```

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